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Learning and the language of thought

Noah D. Goodman
Stanford University

ICCV SIG-11
November 12, 2011

Motivation

What are thoughts made of?
(Concepts... but what are they?)

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What are thoughts made of?
(Concepts... but what are they?)

How are concepts learned?

Statistics and composition

Statistics and composition

Thought is useful
in an uncertain
world

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Why did he yell at me?

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Why did he yell at me?

He wanted to hurt me.
He thought I was a telemarketer.

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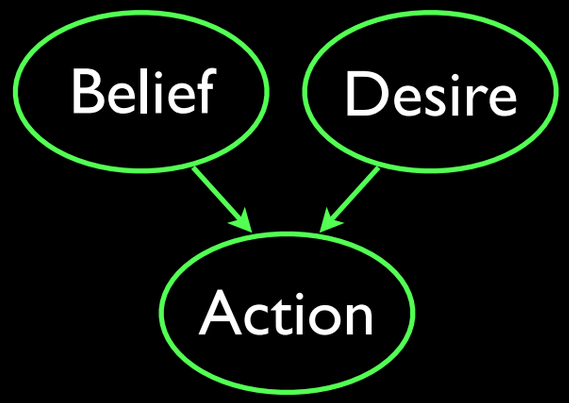

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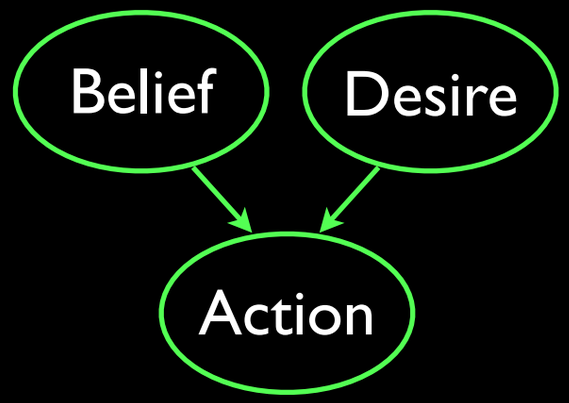

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Probabilistic
inference

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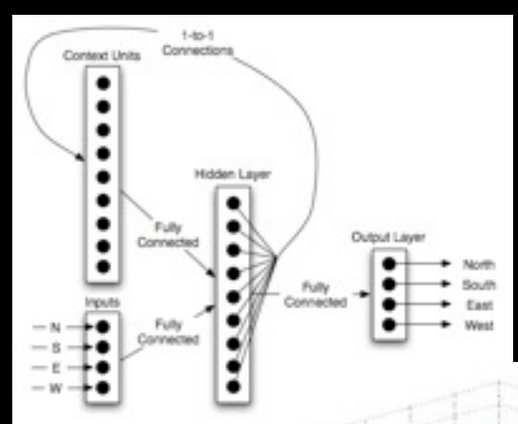

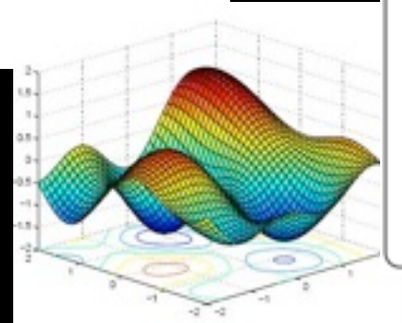

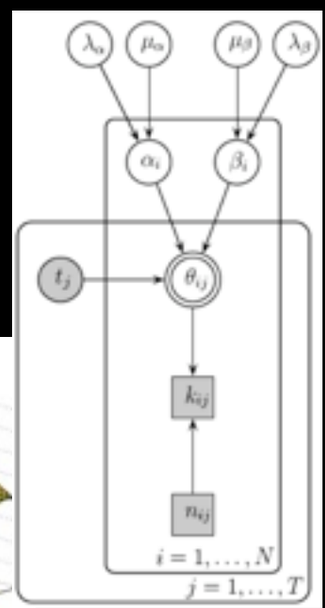

Probabilistic inference

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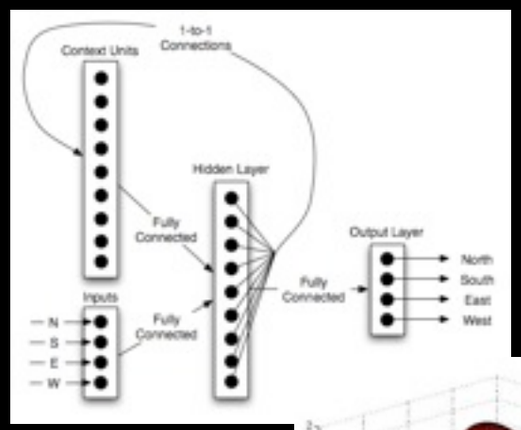

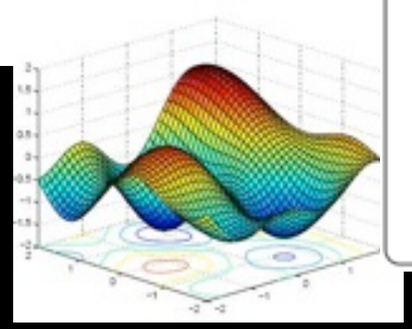

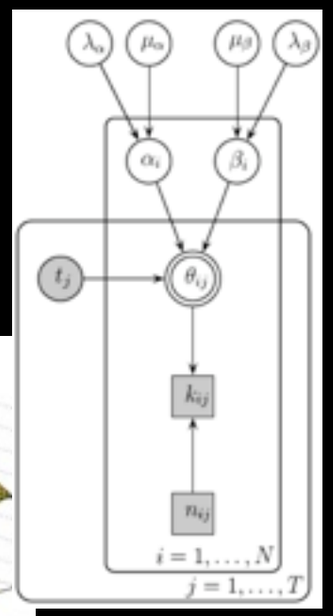

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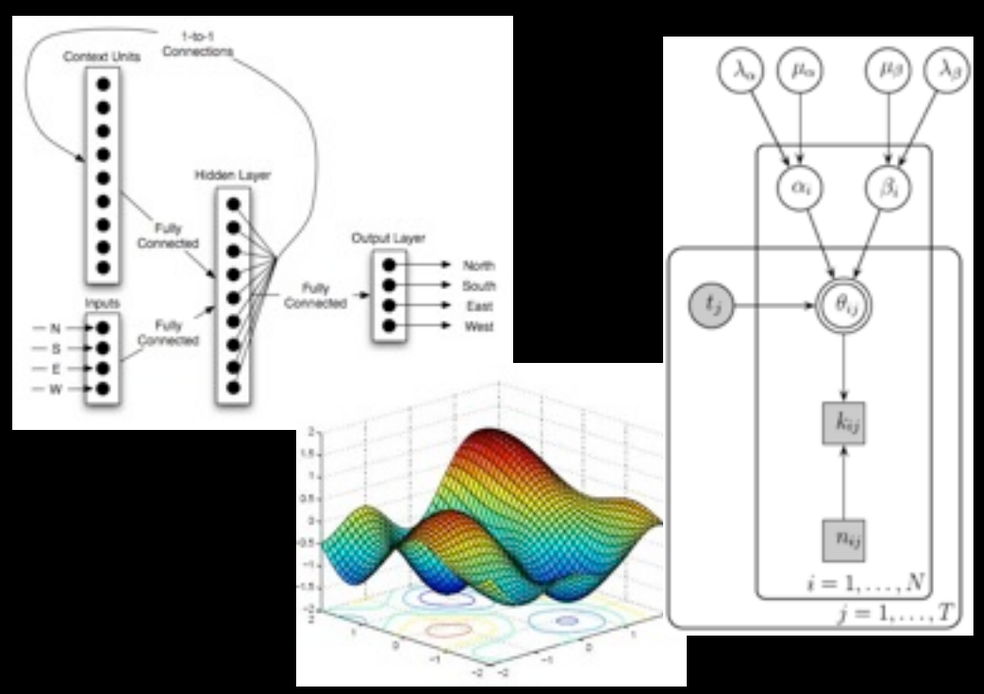

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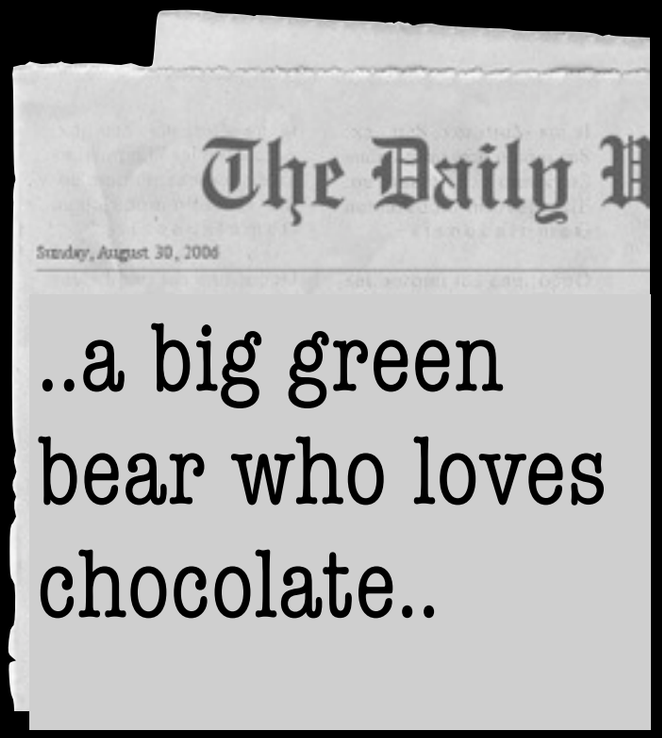

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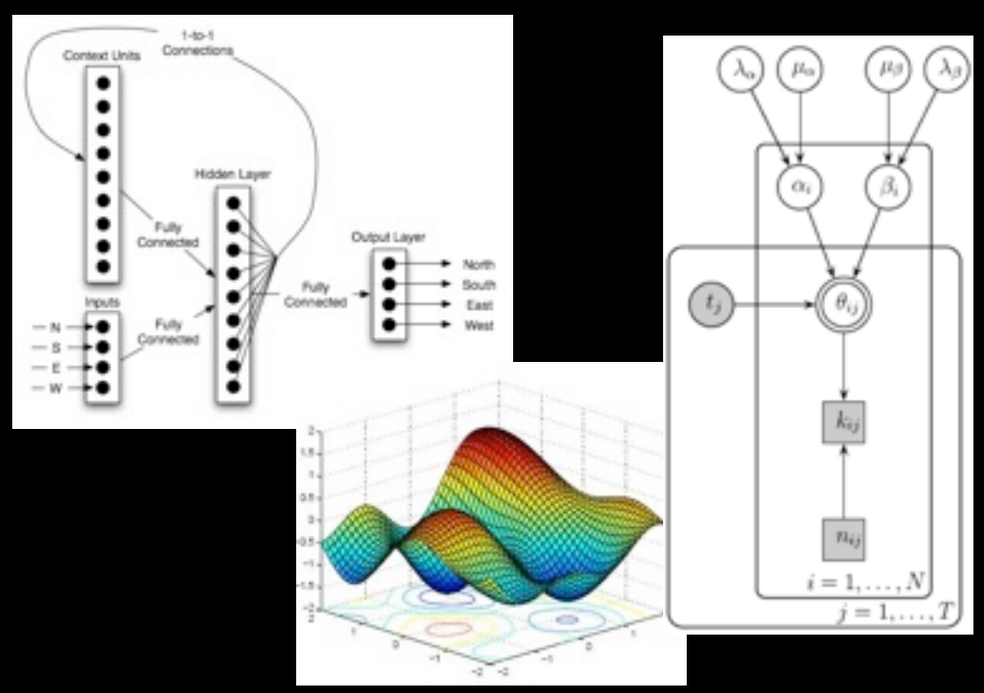

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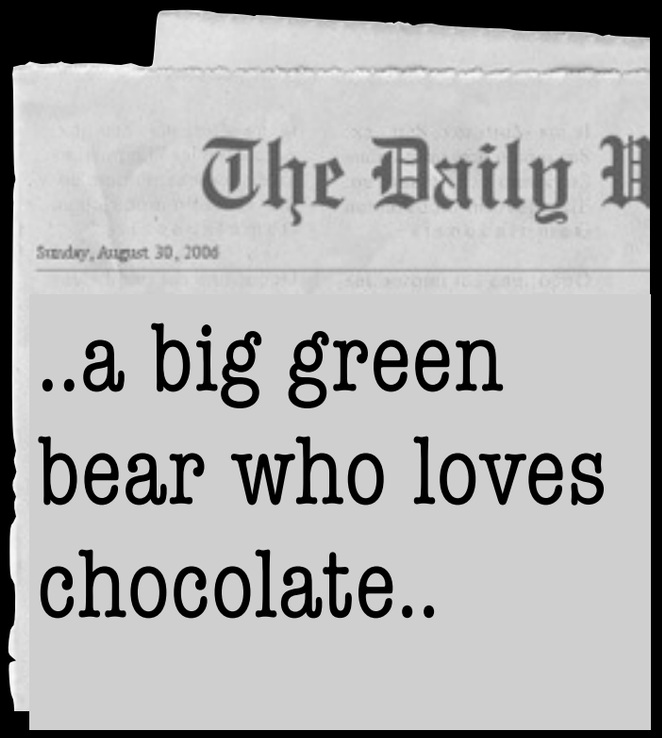

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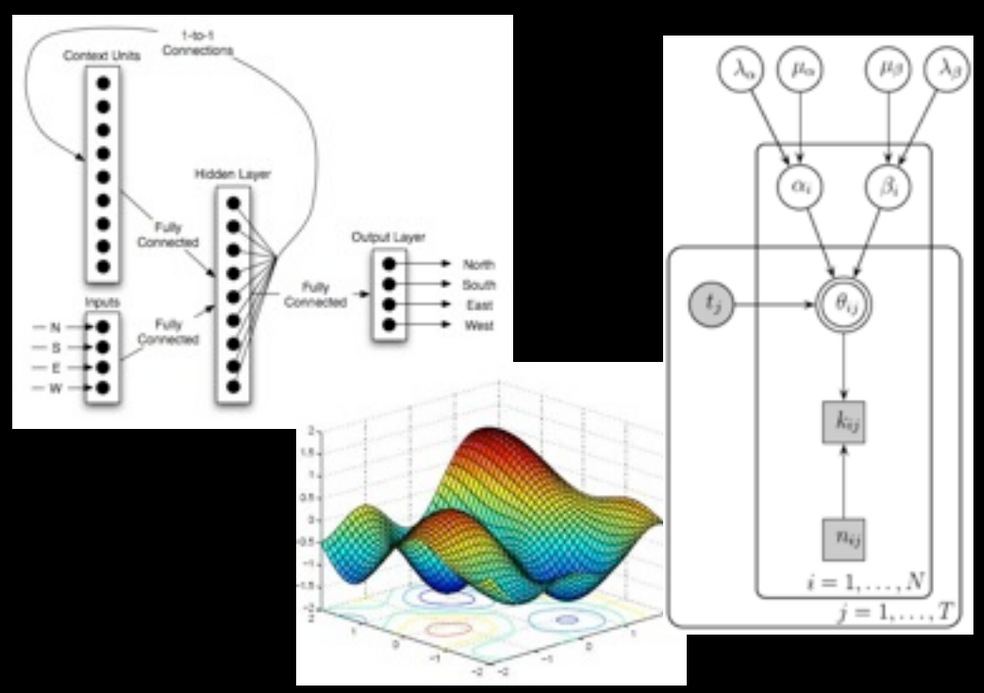

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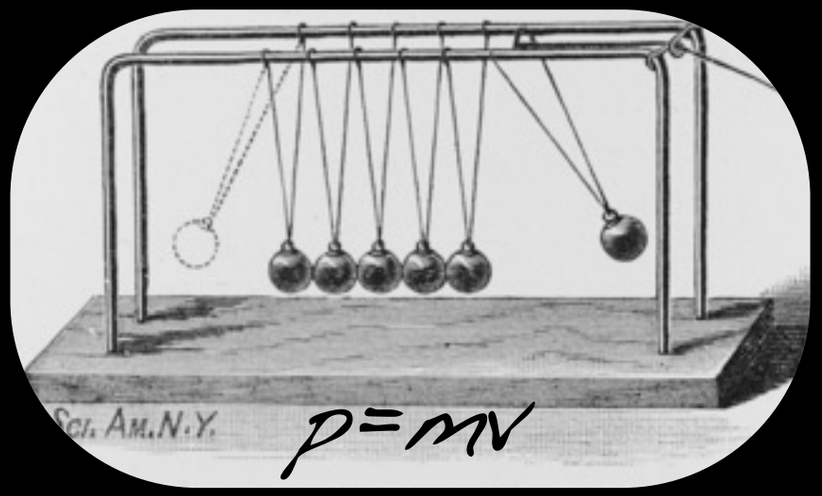

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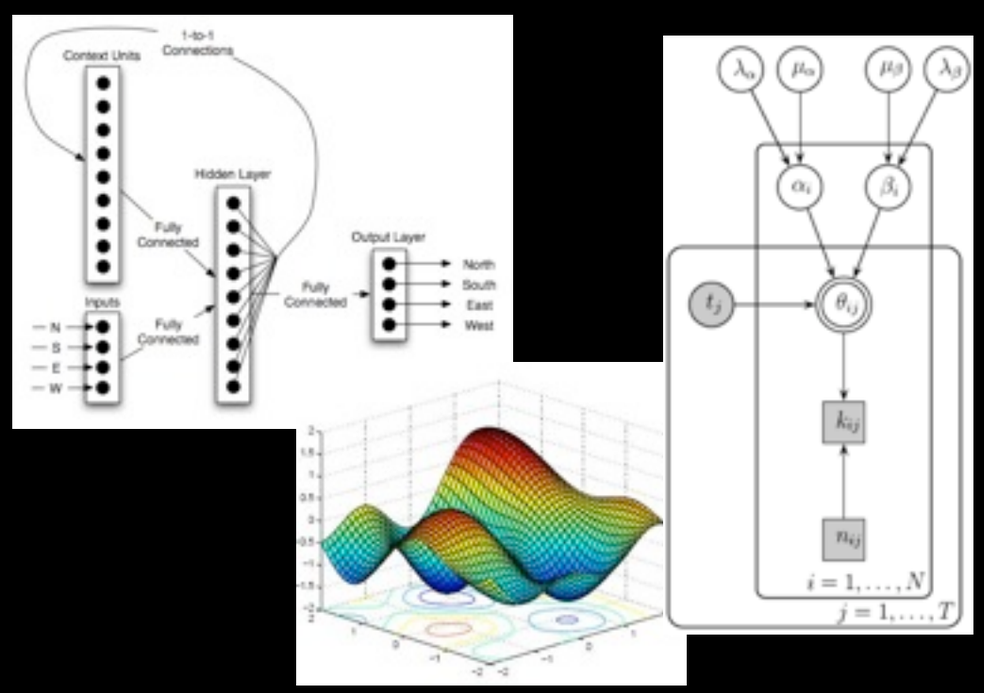

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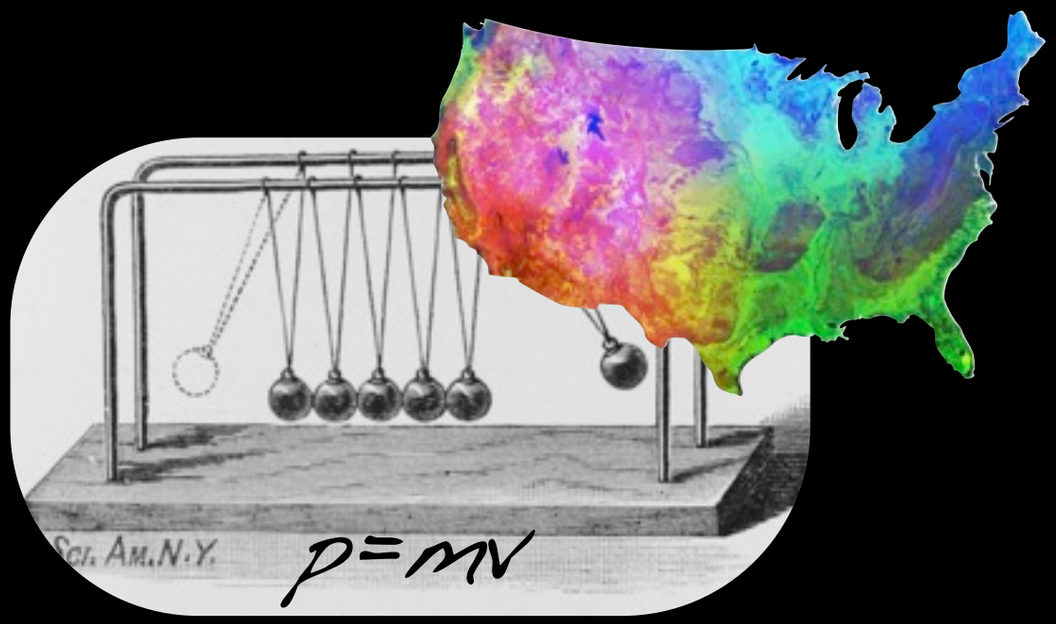

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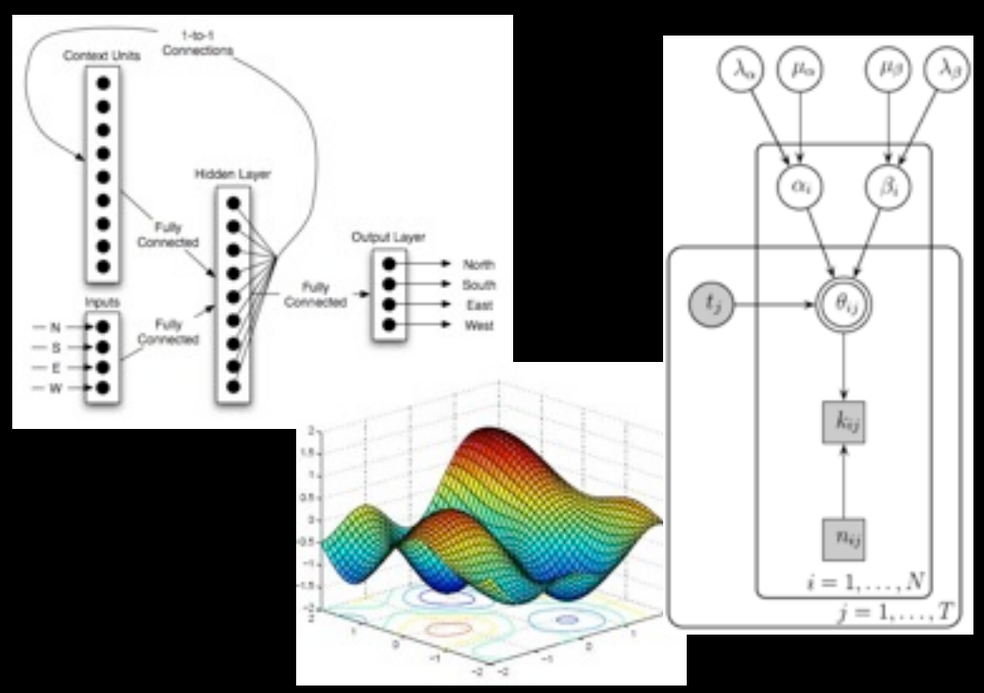

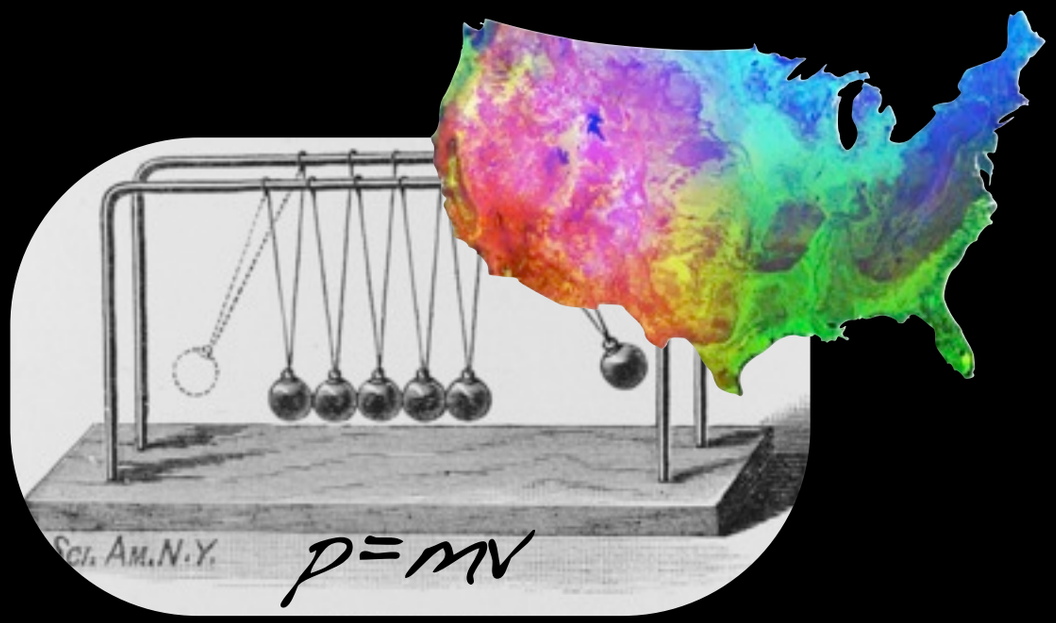

Probabilistic inference

Compositional representations

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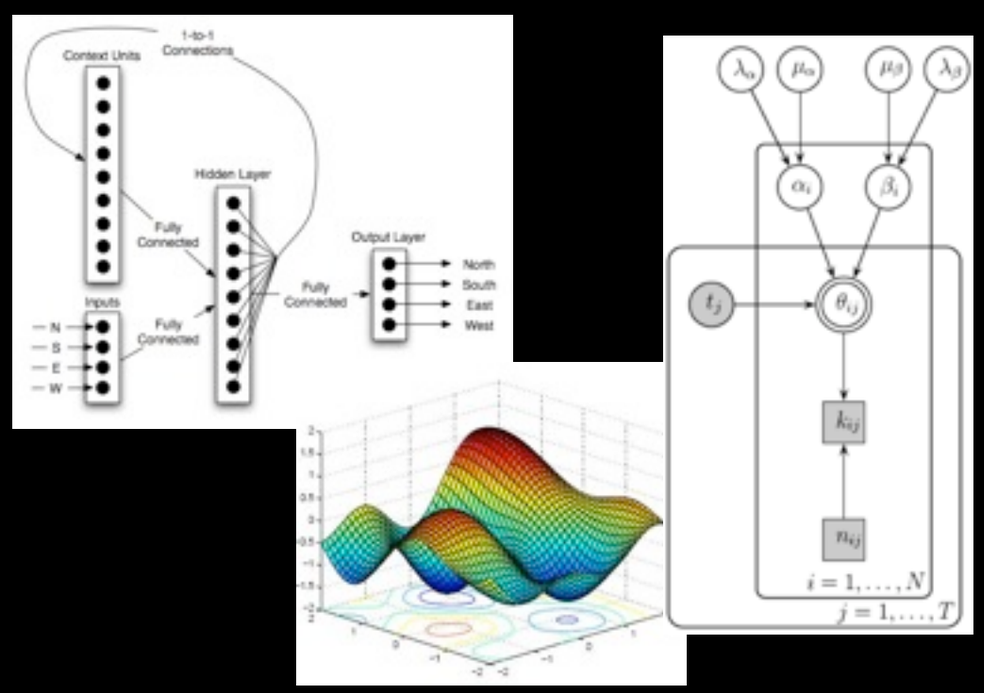

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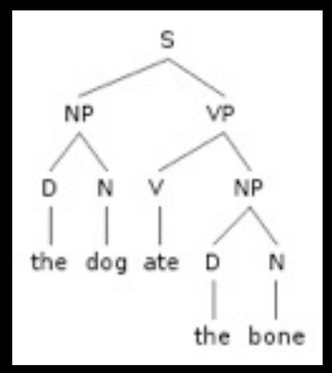

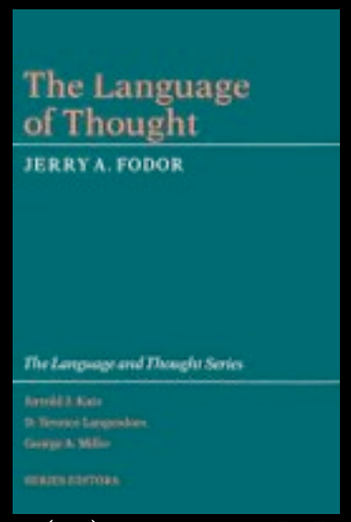

$$\forall x \; King(x) \implies Man(x)$$

$$\forall y \; Man(y) \iff \neg Woman(y)$$

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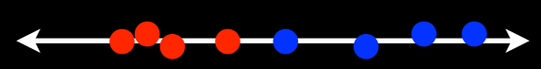

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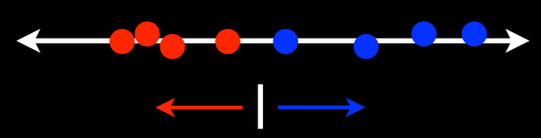

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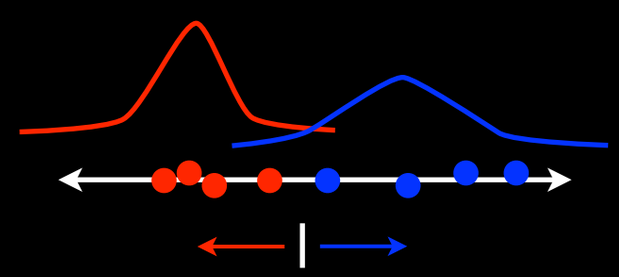

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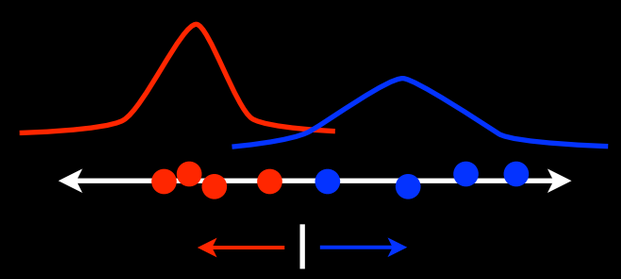

Probabilistic inference

Generative models

Compositional representations

Statistics and composition

Probabilistic language of thought hypothesis

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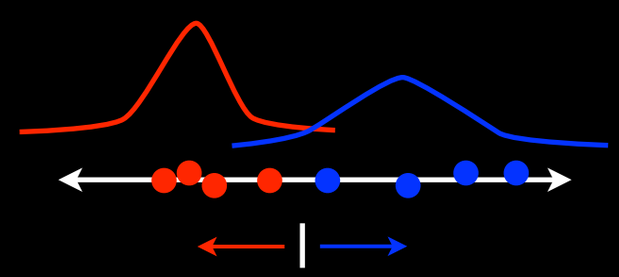

Probabilistic inference

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Probabilistic LoT

- The **probabilistic** language of thought hypothesis:
  - Mental representations are compositional,
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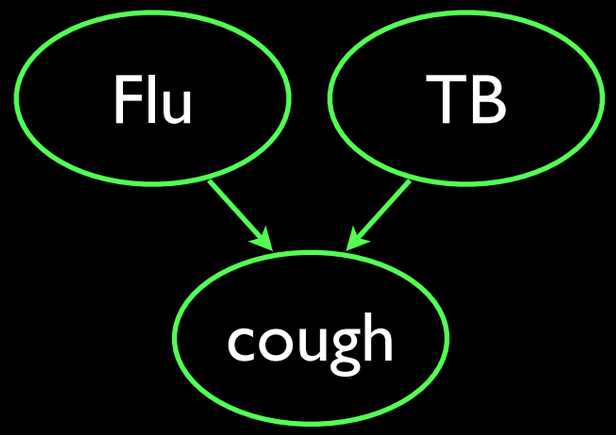
Can this hypothesis be formalized?

Probabilistic generative models



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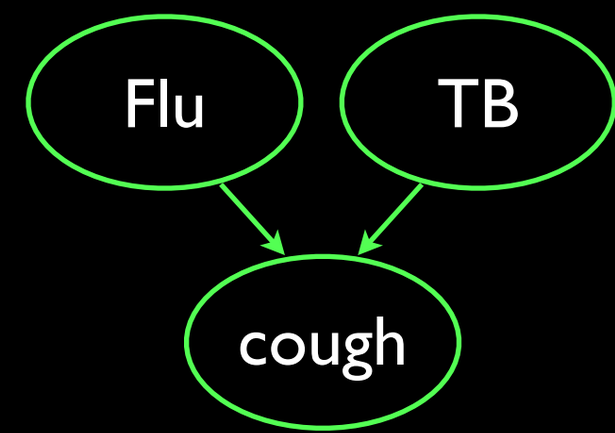

cough	TB	no TB
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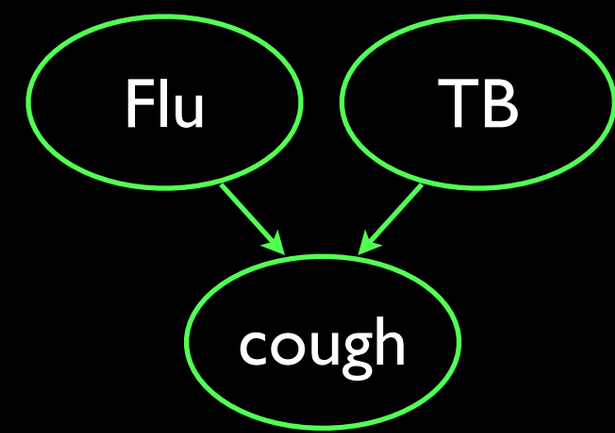

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$$P(H|d) \propto P(d|H)P(H)$$

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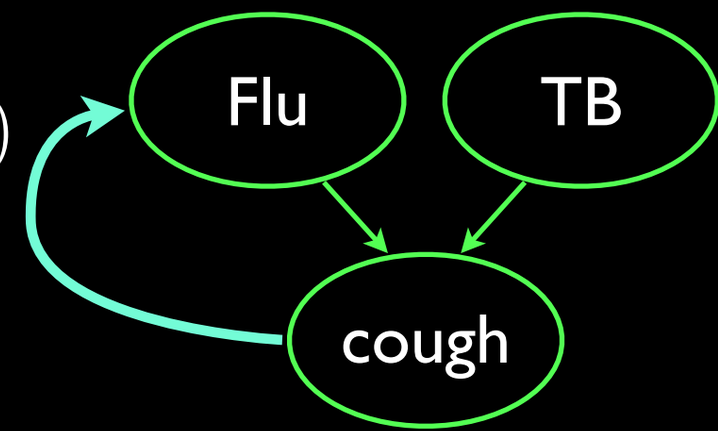

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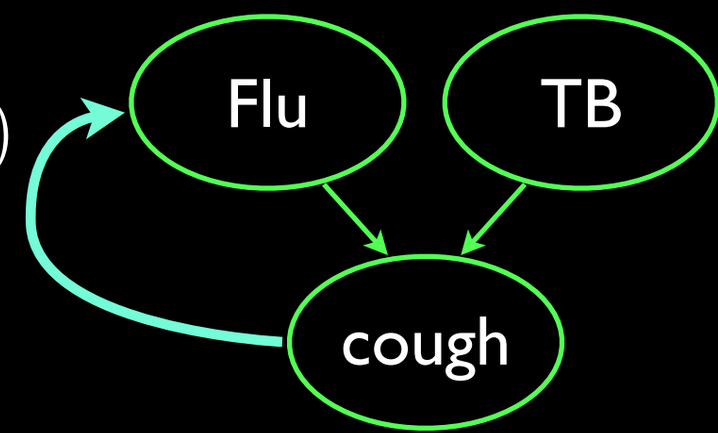

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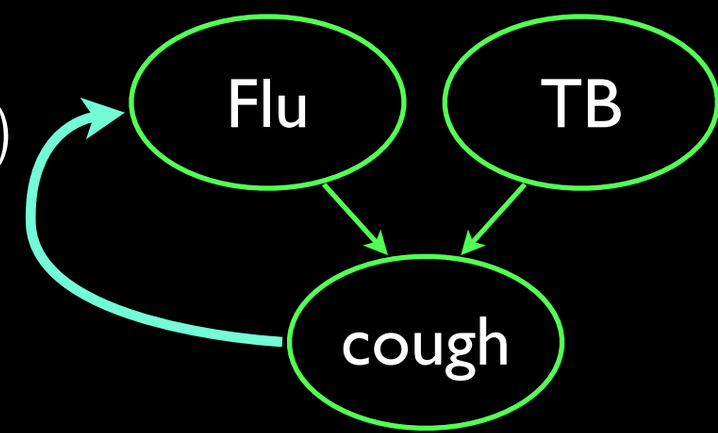

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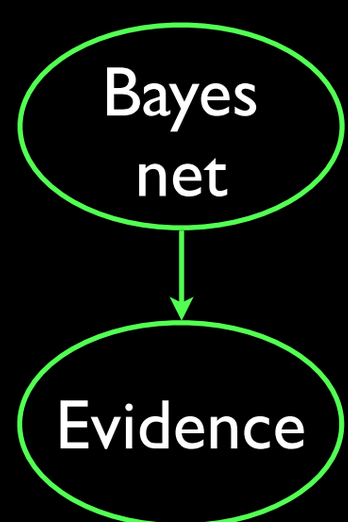

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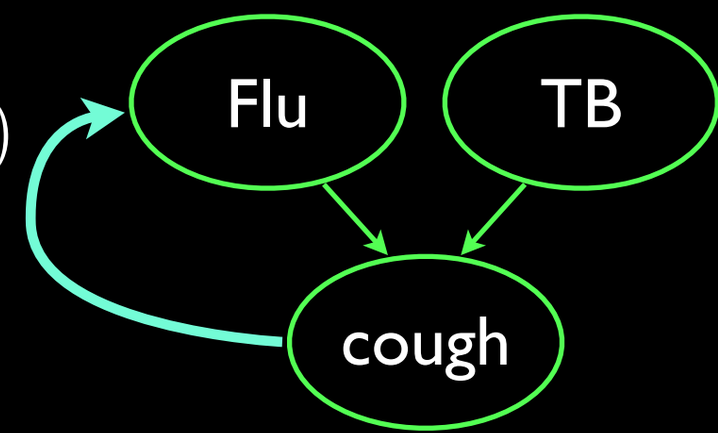

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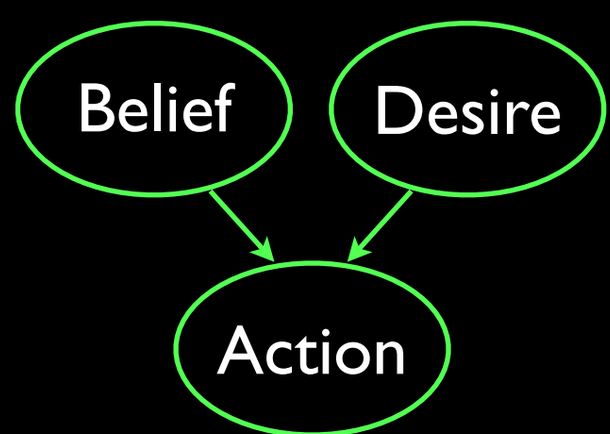

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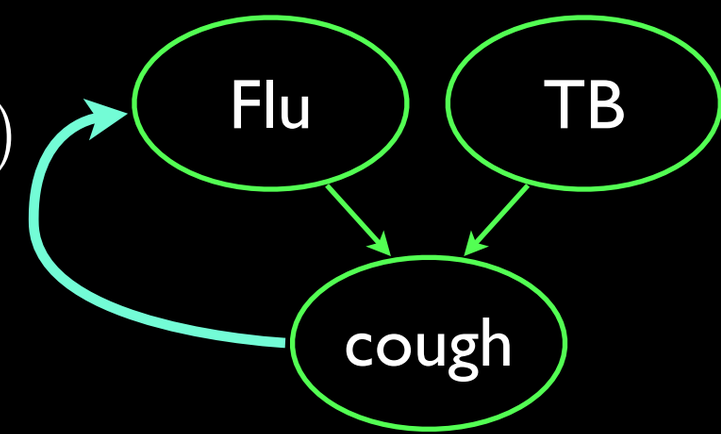

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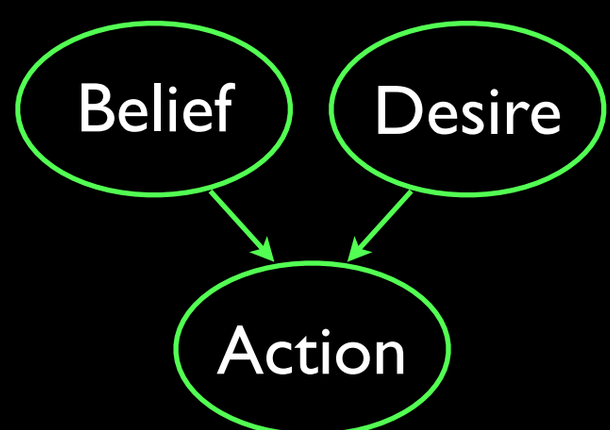

write talk	want fame	want sleep	..
going to workshop	0.7	0.05	
it's raining	0.002	0.004	
:			

λ calculus

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  - Function have parentheses on the wrong side: `(sin x)`
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```
(define 2nd-derivative (repeat derivative))
```

Church

Random primitives:

```
(define a (flip 0.3))
(define b (flip 0.3))
(define c (flip 0.3))
(+ a b c)
```

Goodman, Mansinghka, Roy, Bonawitz, Tenenabum (2008)

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(define a (flip 0.3)) => 1
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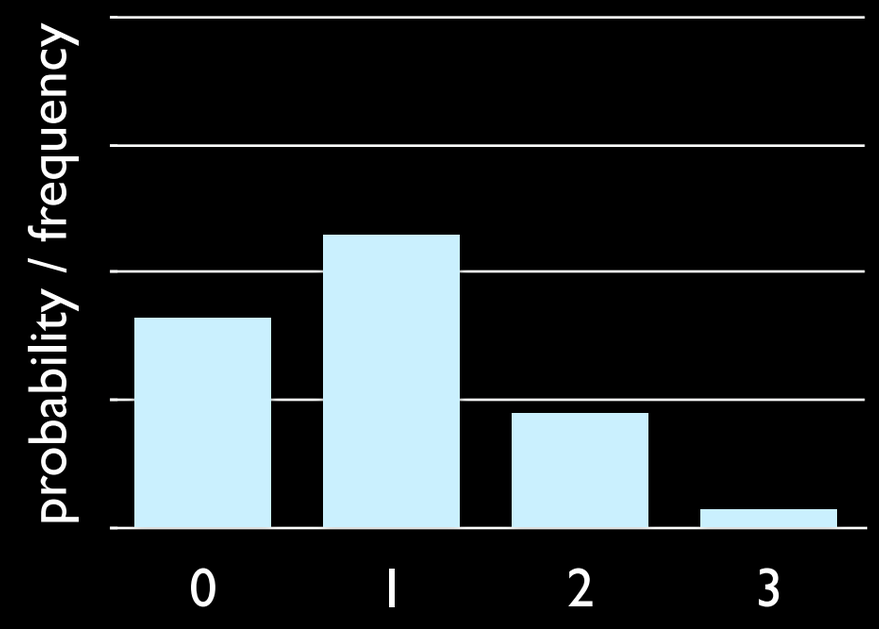

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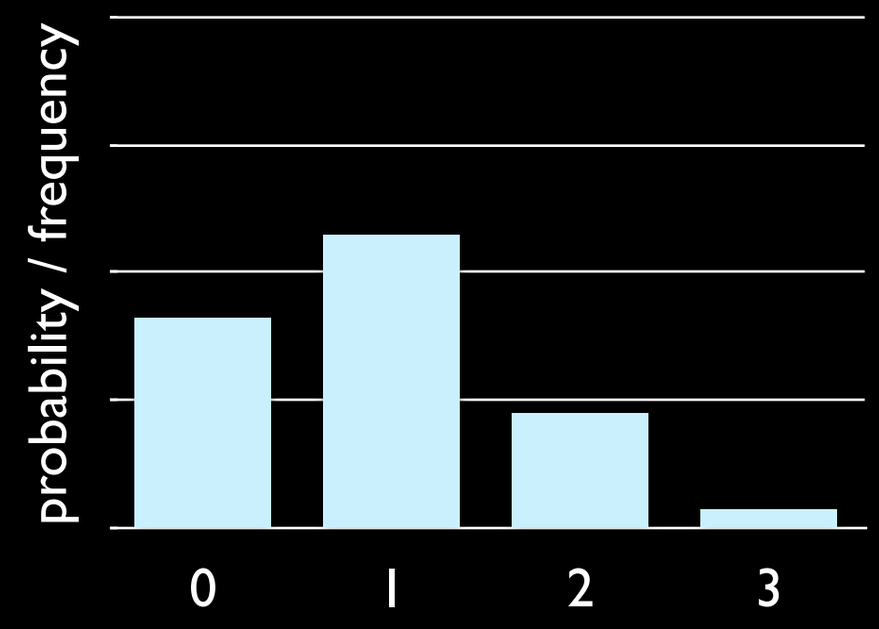
```



Theorem: Any computable distribution can be represented by a Church expression.

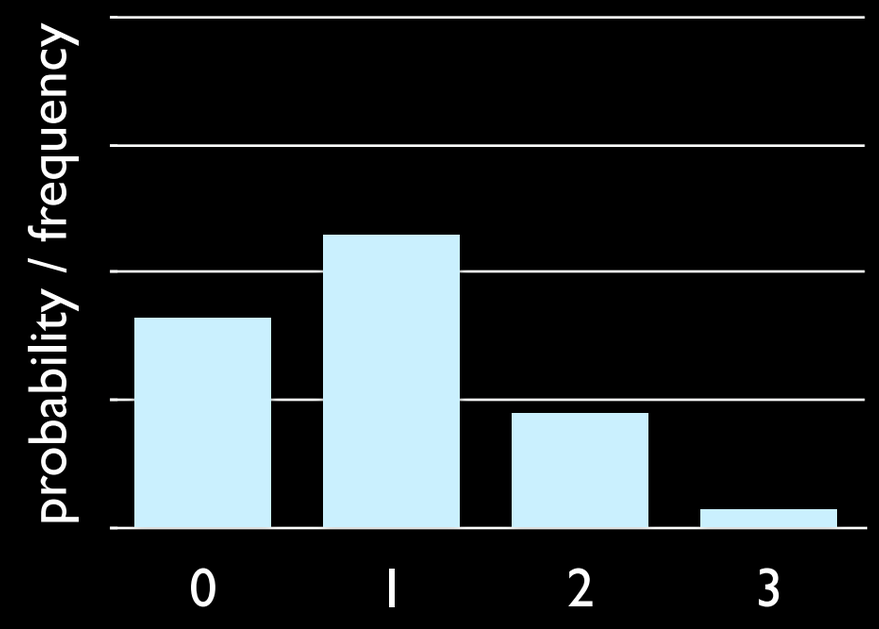
Goodman, Mansinghka, Roy, Bonawitz, Tenenabum (2008)

Church

Random primitives:

```
(define a (flip 0.3))
(define b (flip 0.3))
(define c (flip 0.3))
(+ a b c)
```

```
=> 1 0 0
=> 0 0 0
=> 1 0 1
=> 2 0 1..
```



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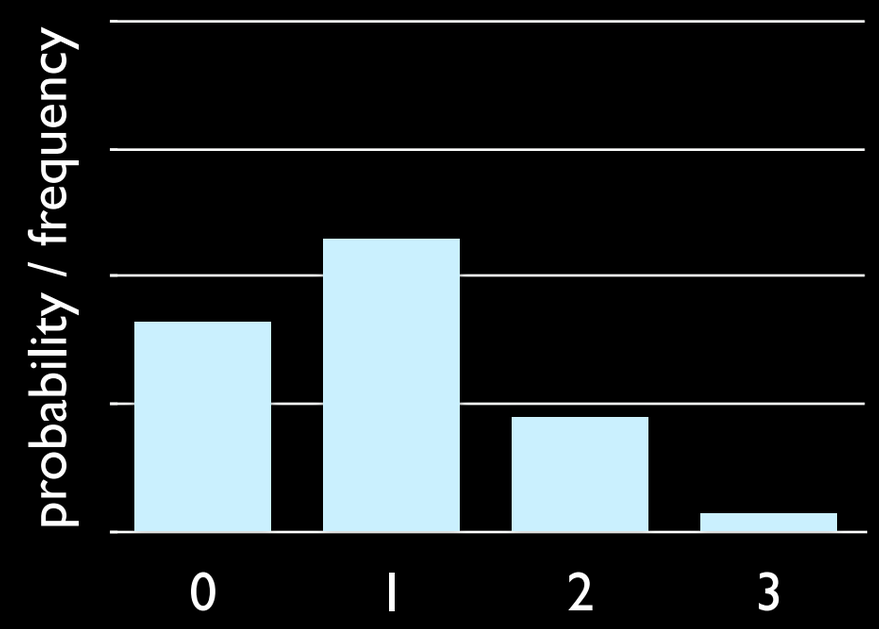

Conditioning (inference):

```
(query
  (define a (flip 0.3))
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Goodman, Mansinghka, Roy, Bonawitz, Tenenabum (2008)

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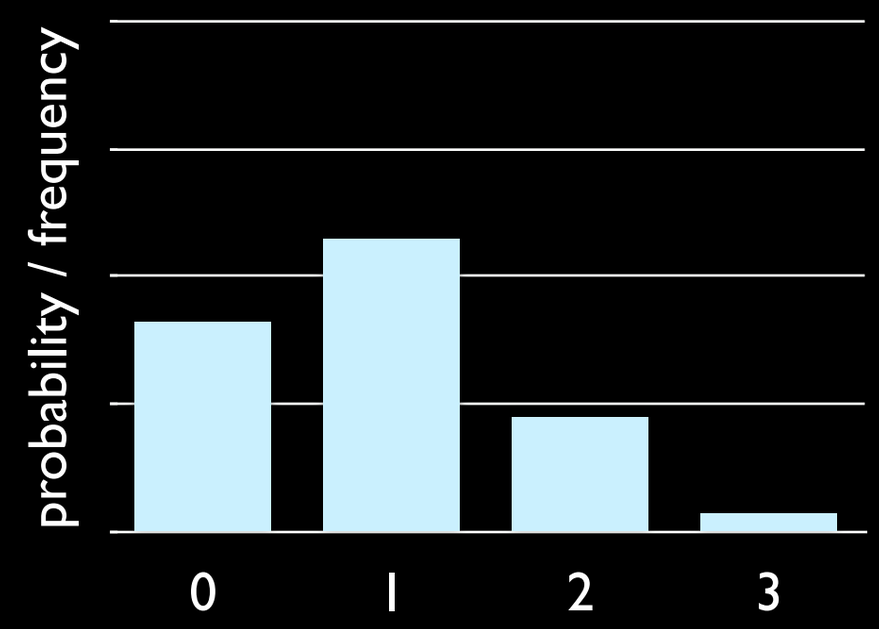

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Goodman, Mansinghka, Roy, Bonawitz, Tenenabum (2008)

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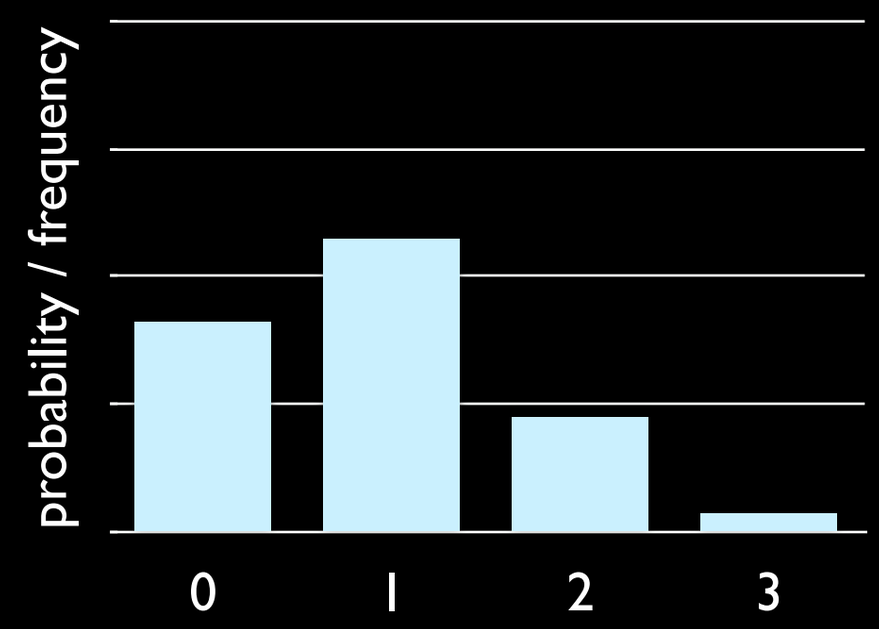

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Condition, must be true

Goodman, Mansinghka, Roy, Bonawitz, Tenenabum (2008)

Church

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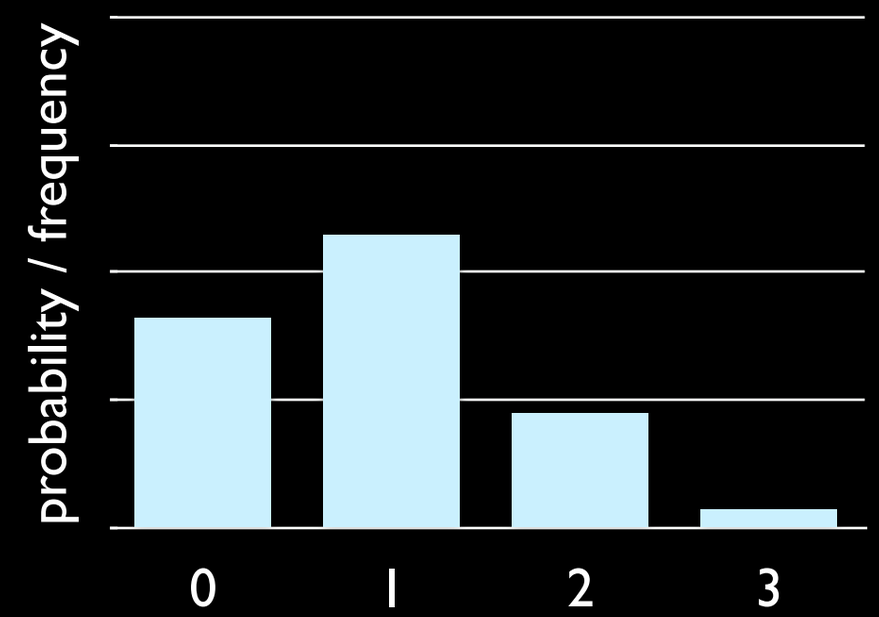

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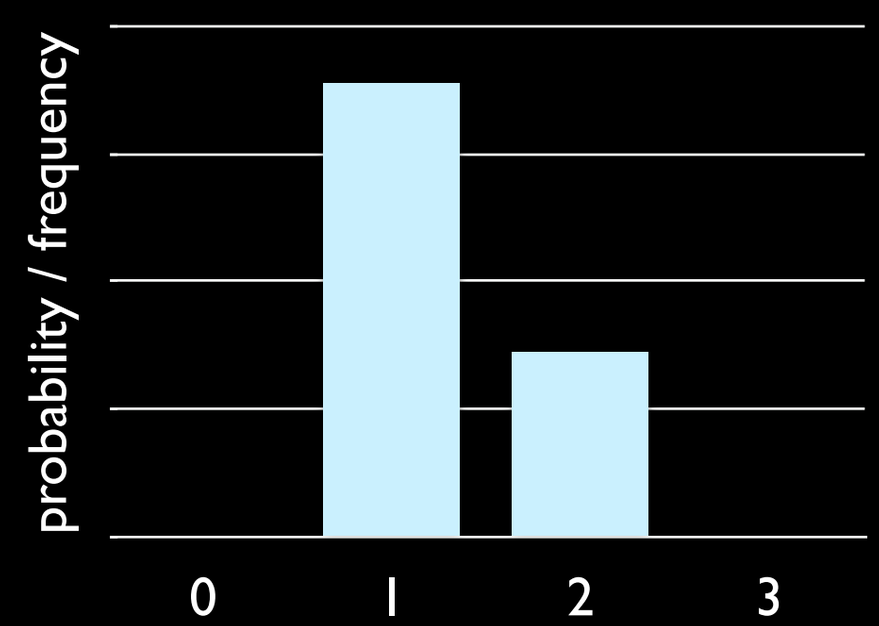


Goodman, Mansinghka, Roy, Bonawitz, Tenenabum (2008)

Example: PCFG

```
(define S (λ () (if (flip)
                    (list (S) (S))
                    ‘s)))
```

Example: PCFG

```
(define S (λ () (if (flip)
                    (list (S) (S))
                    ‘s)))
```

```
(define (tree-unfold symbol)
  (if (terminal? symbol)
      symbol
      (map tree-unfold (transition symbol))))
```

PLoT

- Formalizing the PLoT:
  Mental representations (concepts) are functions in a **stochastic** lambda calculus (e.g. Church).
  - Separate the process of inference from representations and the inferences they license (Cf. Marr's levels).
  - (However see our work on inference....)

http://projects.csail.mit.edu/church

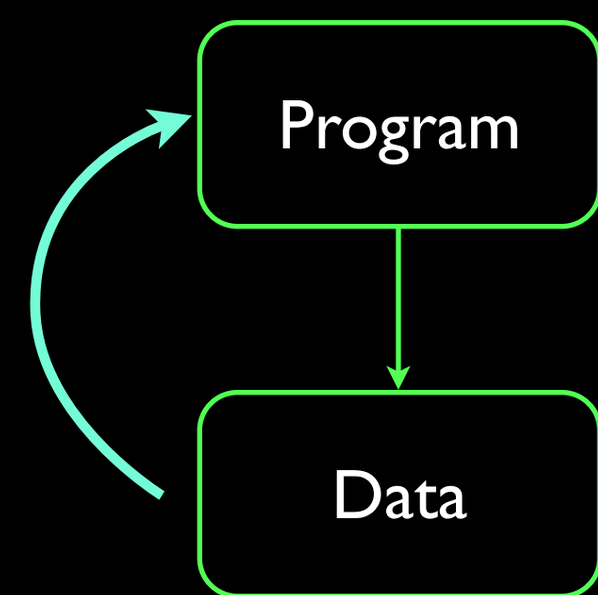
Outline

If concepts are probabilistic programs, then concept learning is *probabilistic program induction*.

Outline

If concepts are probabilistic programs, then concept learning is *probabilistic program induction*.

```
(query
 (define concept (sample-PLoT-expression))
 concept
 (and (= (noisy (sample concept)) obs1)
      (= (noisy (sample concept)) obs2)
      ...))
```



Categorization

Medin & Schaffer (1978):

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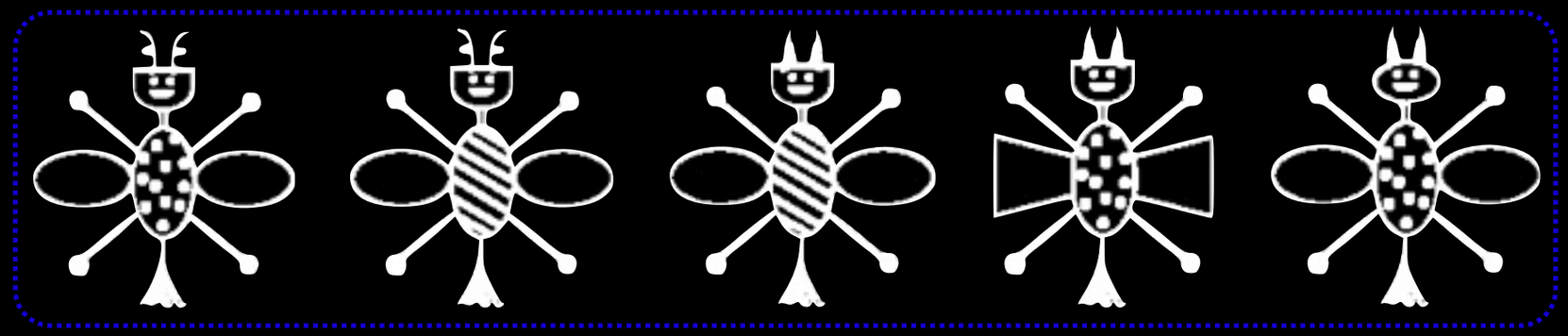

"These are **Feps**"

Categorization

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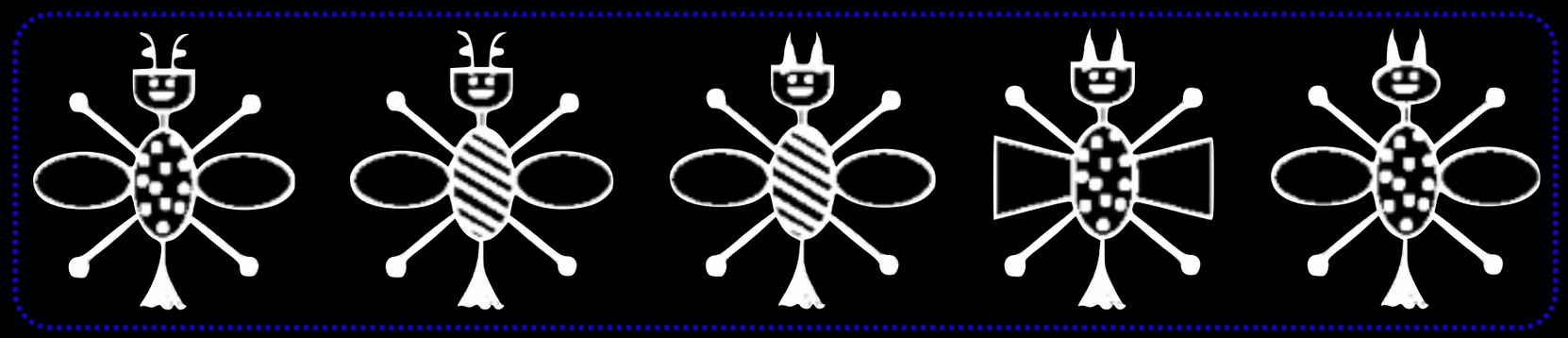

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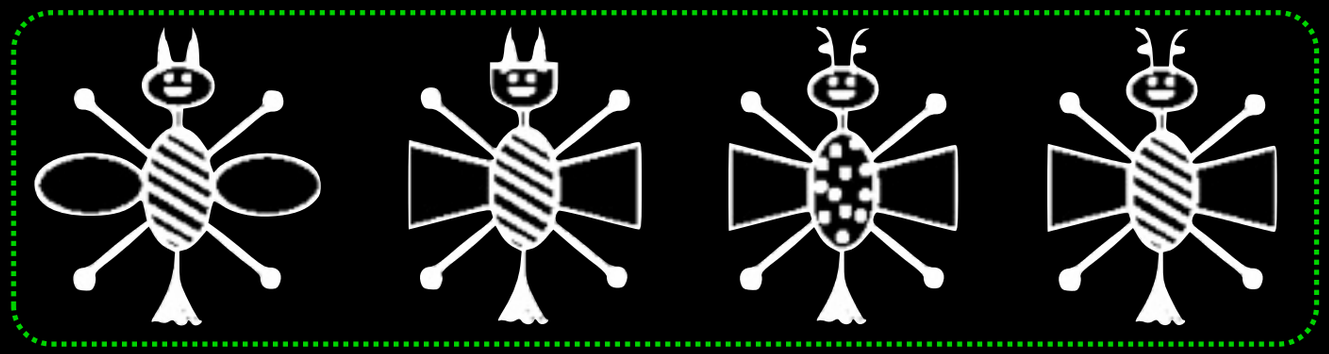

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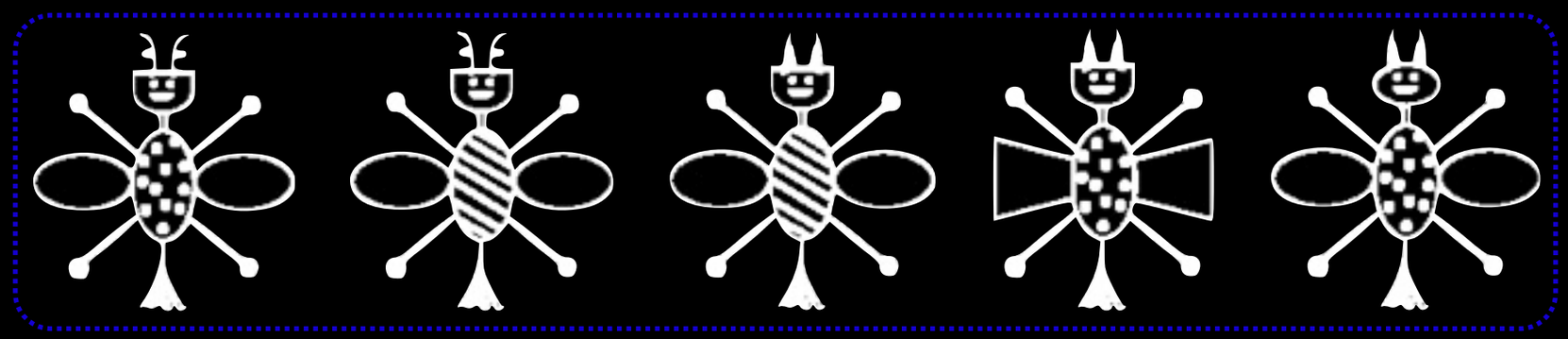

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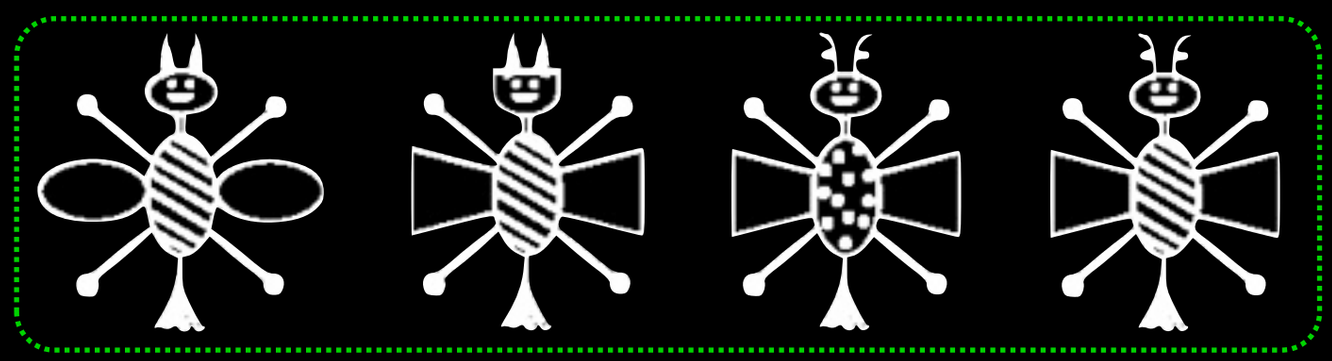

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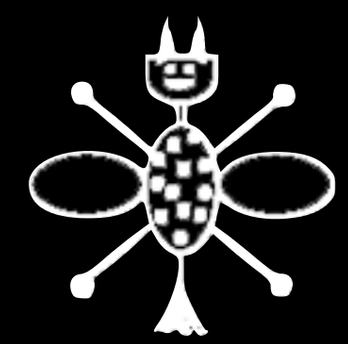
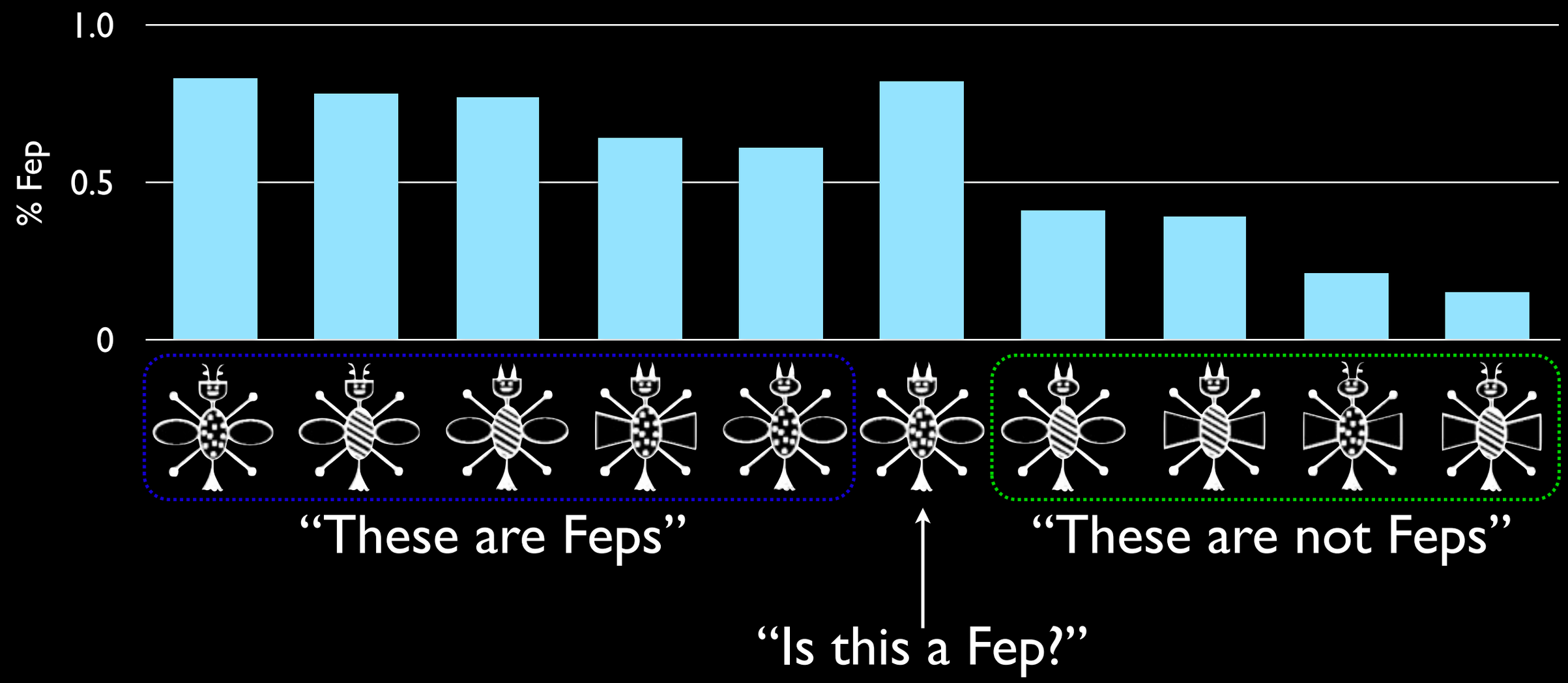


"Is this a Fep?"

Categorization

Medin & Schaffer, 1978 (data from Nosofsky, et al.,1994):



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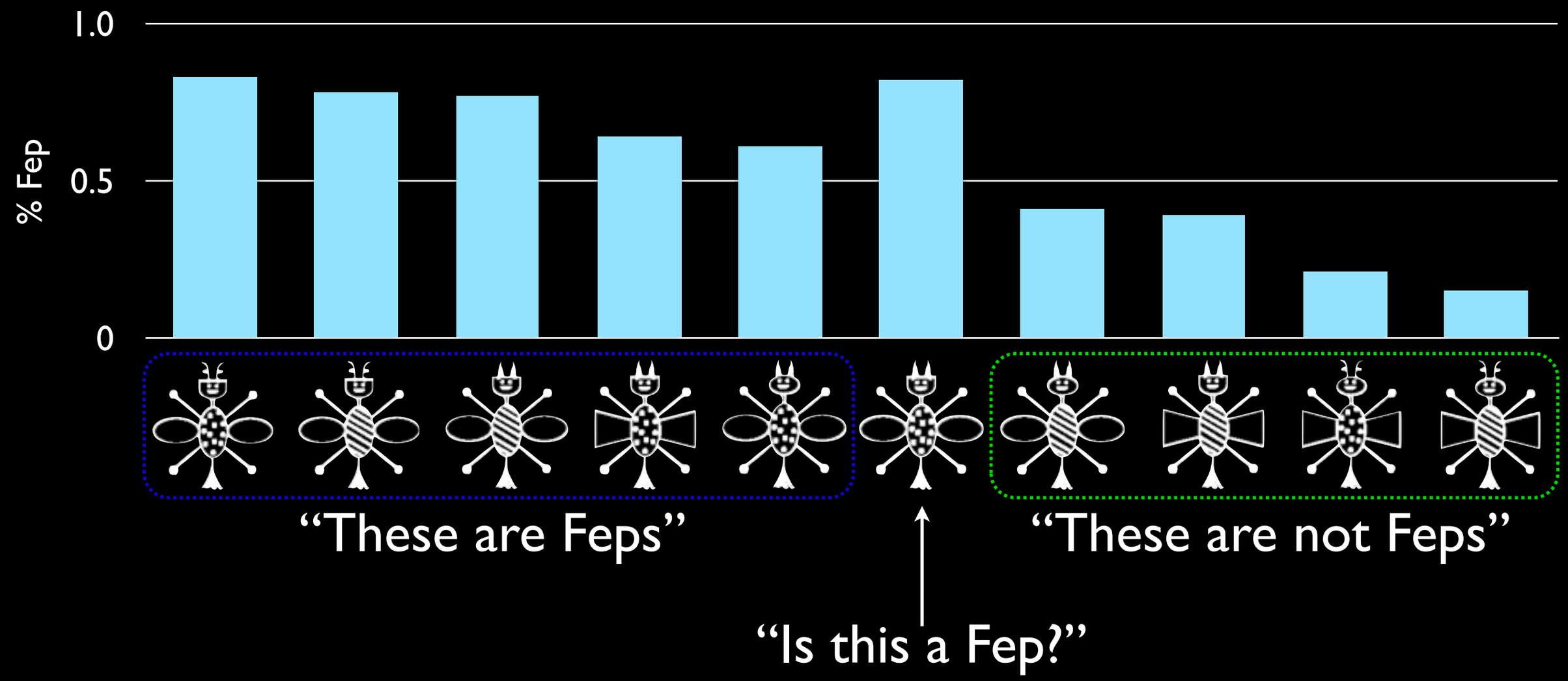
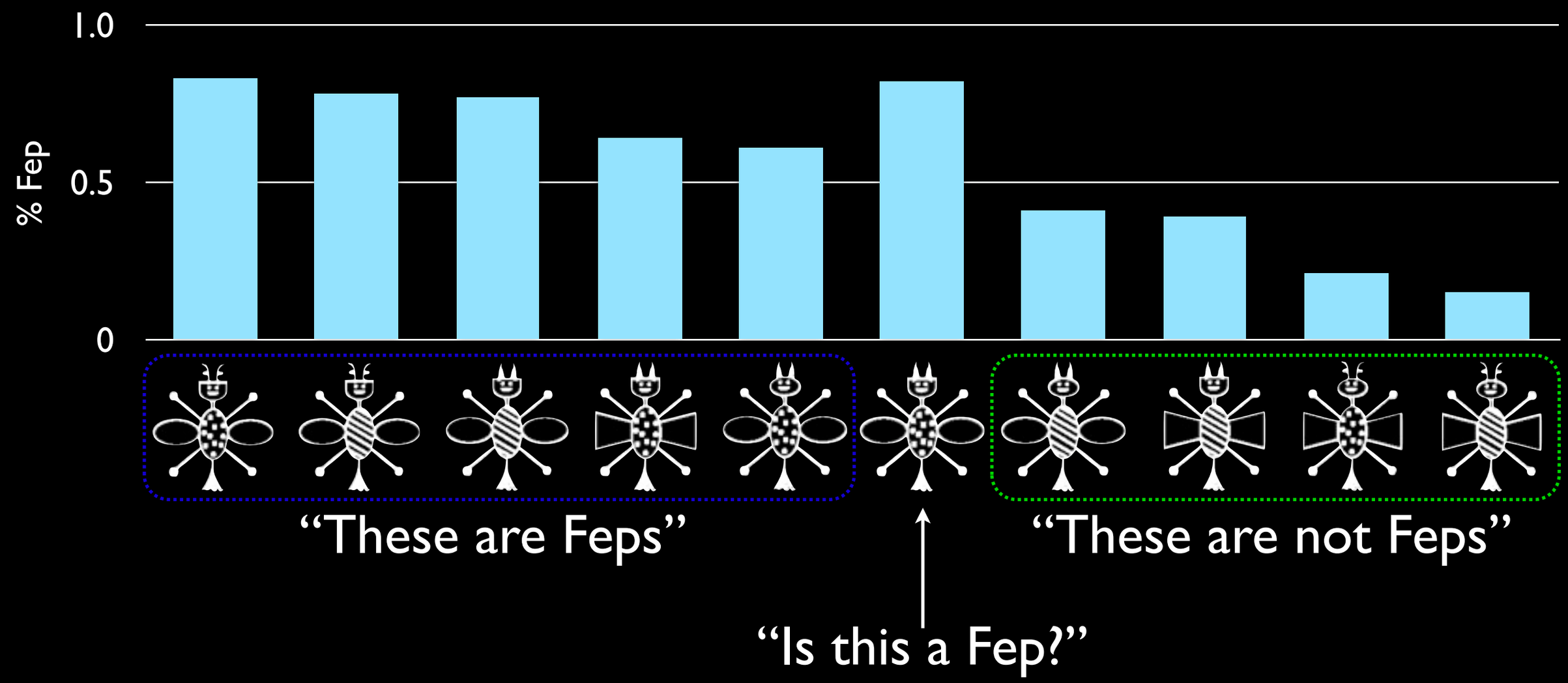


- Graded judgements

Categorization

Medin & Schaffer, 1978 (data from Nosofsky, et al.,1994):



- Graded judgements
- Typicality

Categorization

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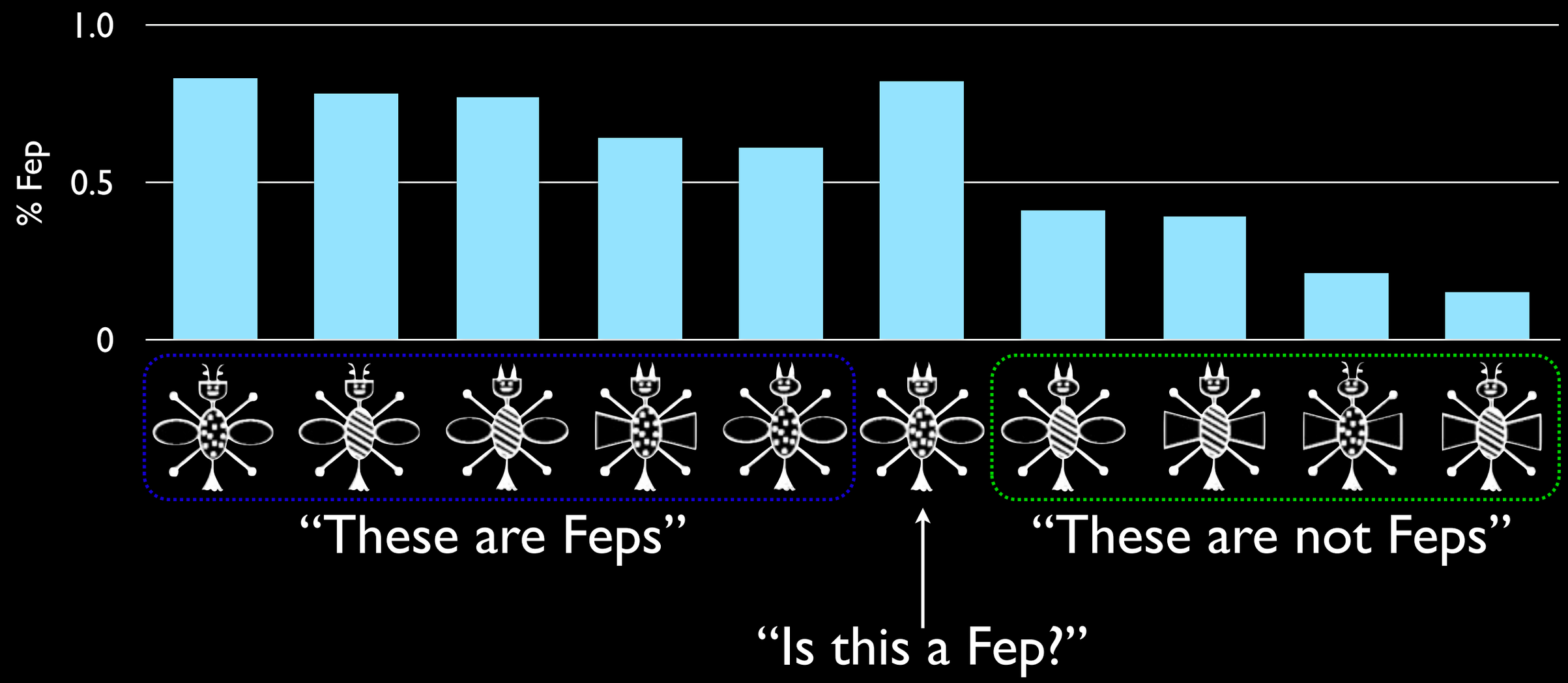
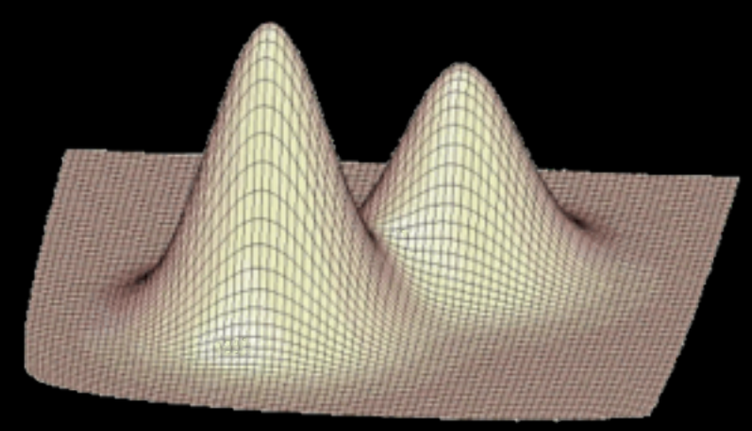


- Graded judgements
- Typicality
- Prototype enhancement

Categorization

- Rule-based category learning:
  - Infinitely many concepts formed compositionally.

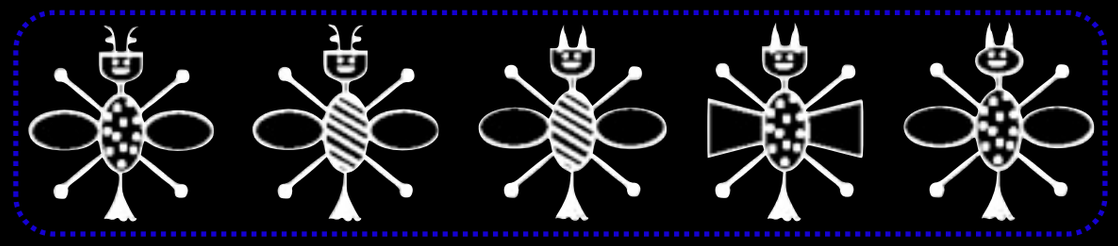
- Statistical category learning:
  - Graded inferences from sparse, noisy evidence.



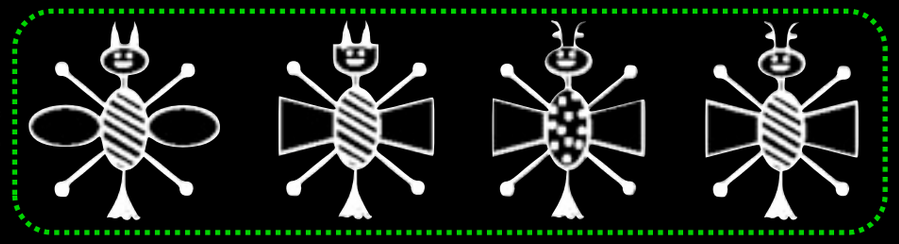
Generating rules

"It's a Fep if it has flat head and round wings"

Feps:



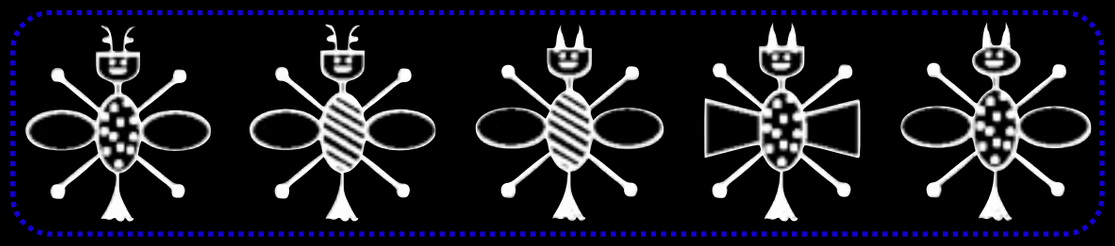
non-Feps:



Generating rules

```
(define fep?
 (λ (x)
  (and (flat-head x)
       (round-wings x))))
```

Feps:



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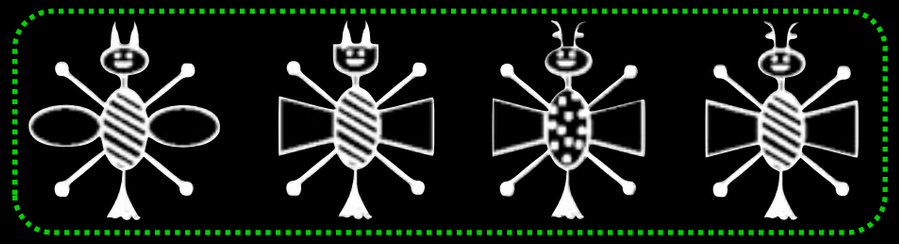

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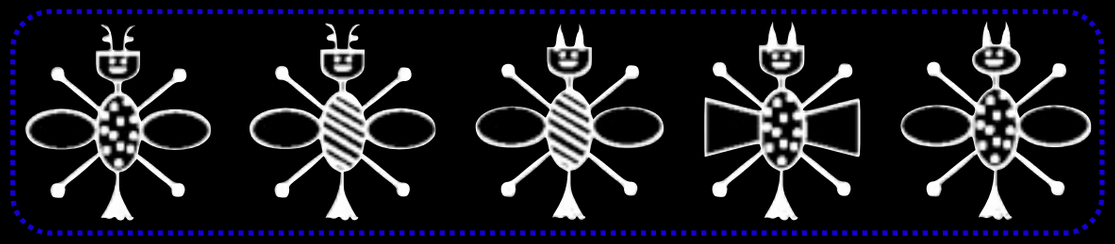

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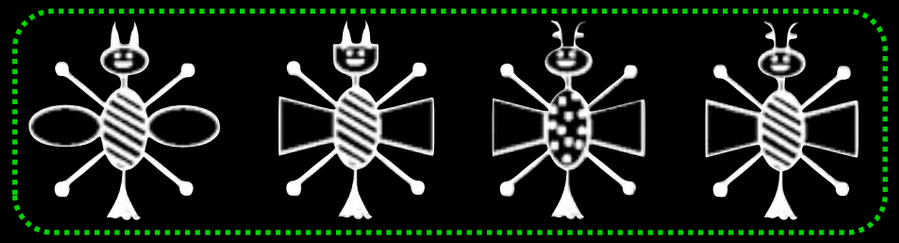
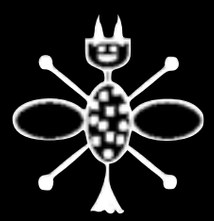


(fep? )

=> true

Generating rules

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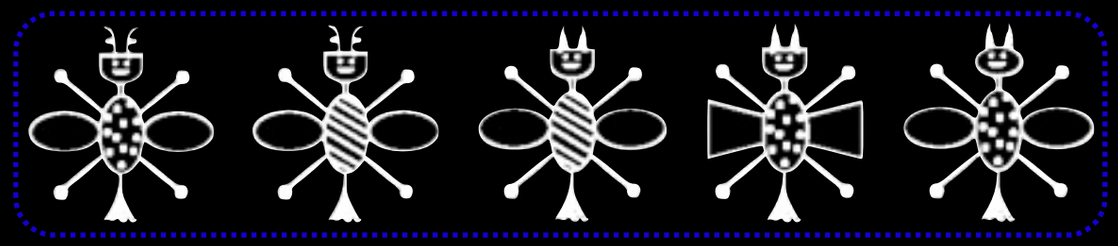
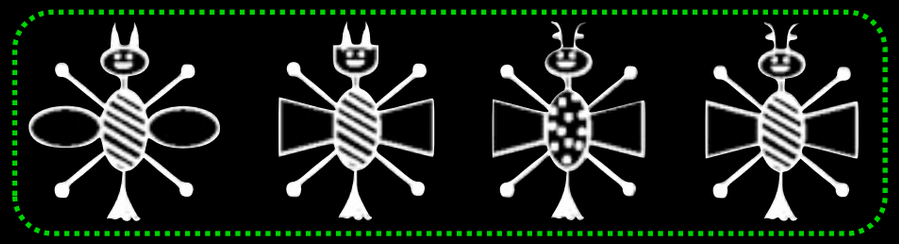
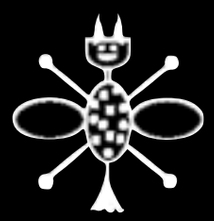
```

Feps: 

non-Feps: 

(fep? )

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```
(define rule-generator
 (λ ()
  (if (flip 0.3)
      (sample-feature)
      (combine-rules (sample-feature)
                     (rule-generator)))))
```

```
(define combine-rules
 (λ (r1 r2)
  (λ (x) (and (r1 x) (r2 x)))))
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Generating rules

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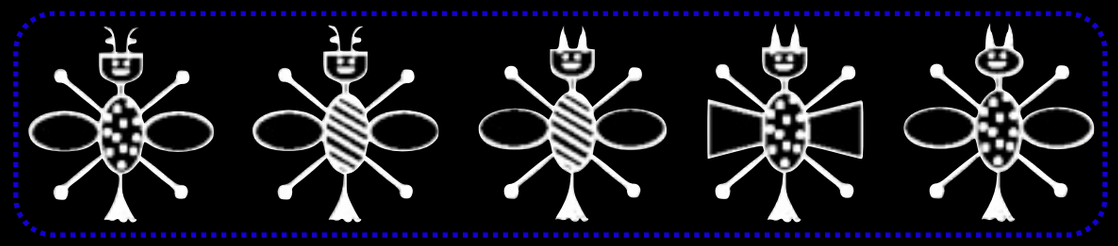
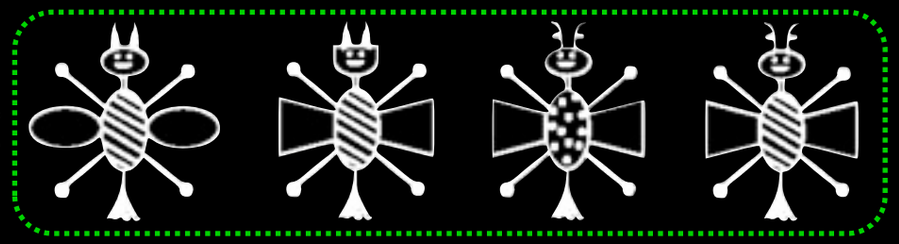
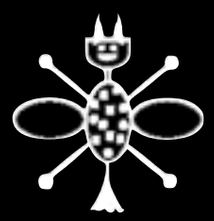
Feps: 

non-Feps: 

(fep? )

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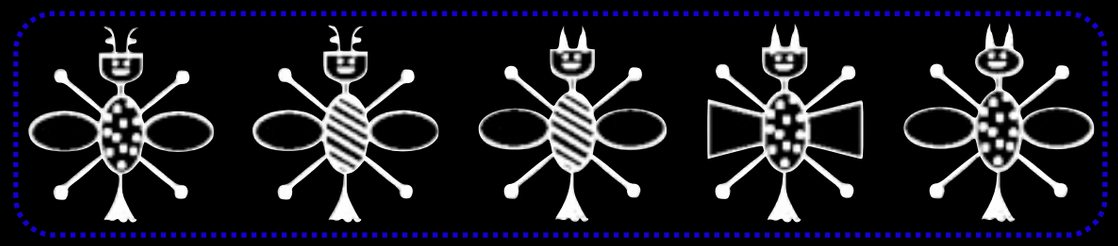

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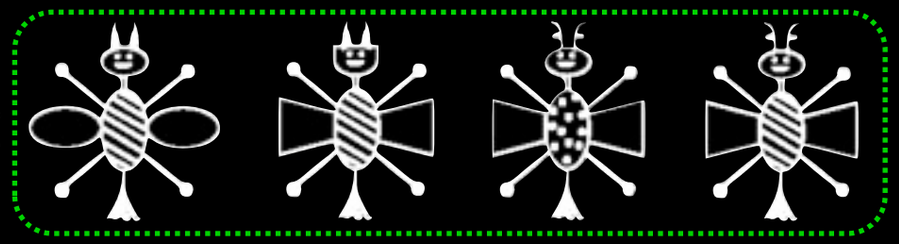
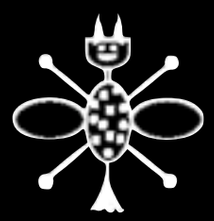


(fep? )

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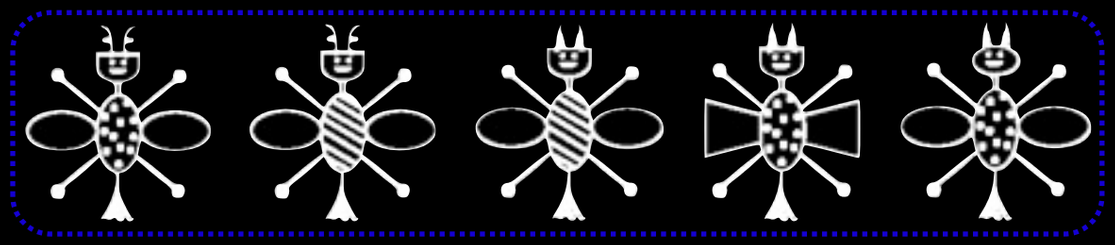
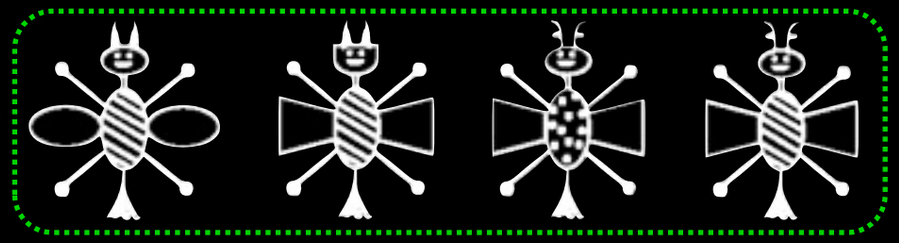
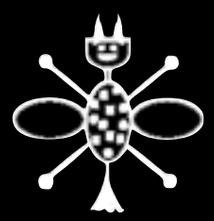
Feps: 

non-Feps: 

(fep? )

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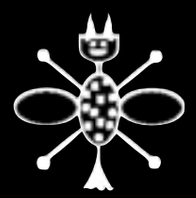
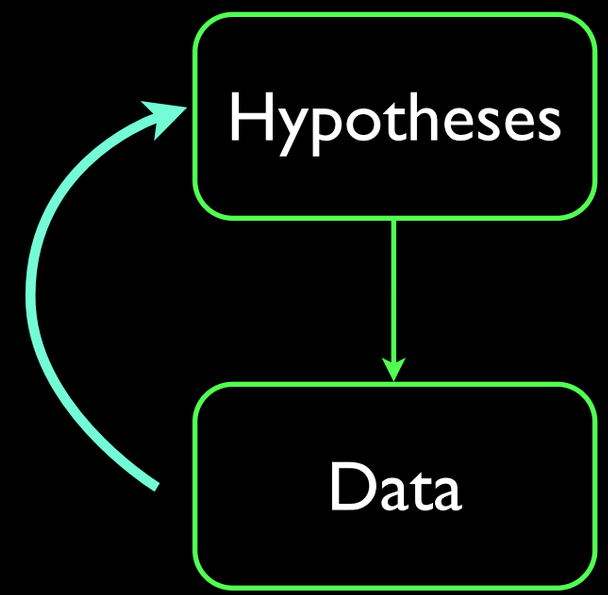
```

```
(define combine-rules
 (λ (r1 r2)
  (λ (x) (and (r1 x) (r2 x)))))
```

- Longer rules have lower probability (Occam's razor).

Generating rules

Put uncertainty over rule probabilities:

```
(define rule-prob (uniform 0 1))
(define rule-generator
 (λ ()
  (if (flip rule-prob)
      ...
```

Generate disjunctive normal form (DNF) rules:

```
(define fep?
 (λ (x)
  (and (or (flat-head x) ...)
       (or (round-wings x) ...)
       ...)))
```

The general idea:
grammar-based induction.

Inducing rules

Inference:

```
(query
 (define rule (rule-generator))
 (rule  )
 (and (= (rule  ) true)
      (= (rule  ) false)
      ...))
```



Inducing rules

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```
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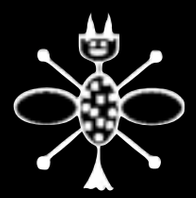
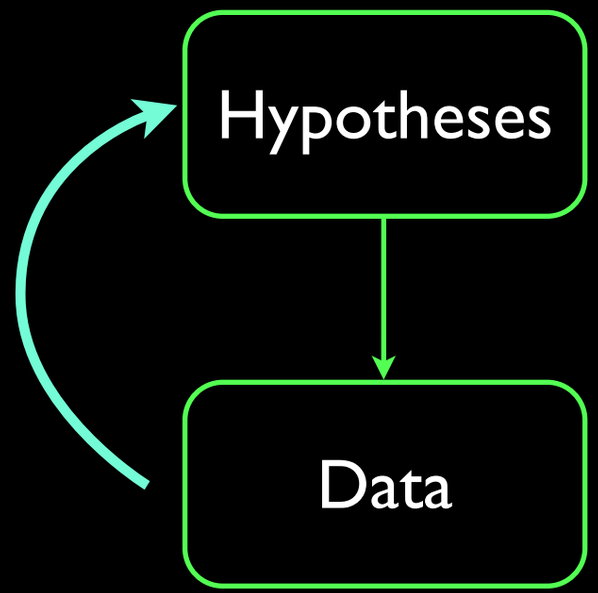
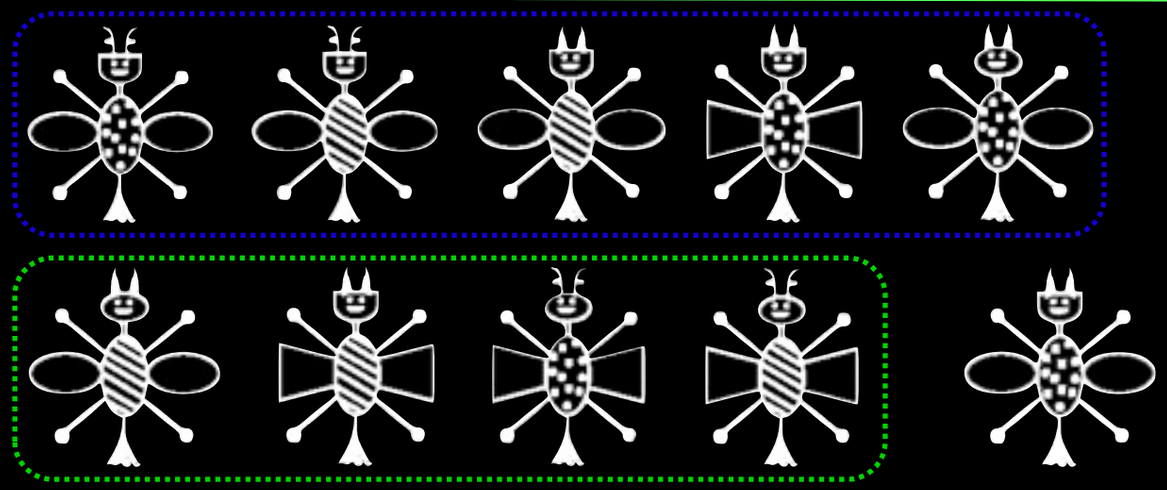
 (define rule (rule-generator))
 (rule  )
 (and (= (noisy (rule  )) true)
      (= (noisy (rule  )) false)
      ...))
```

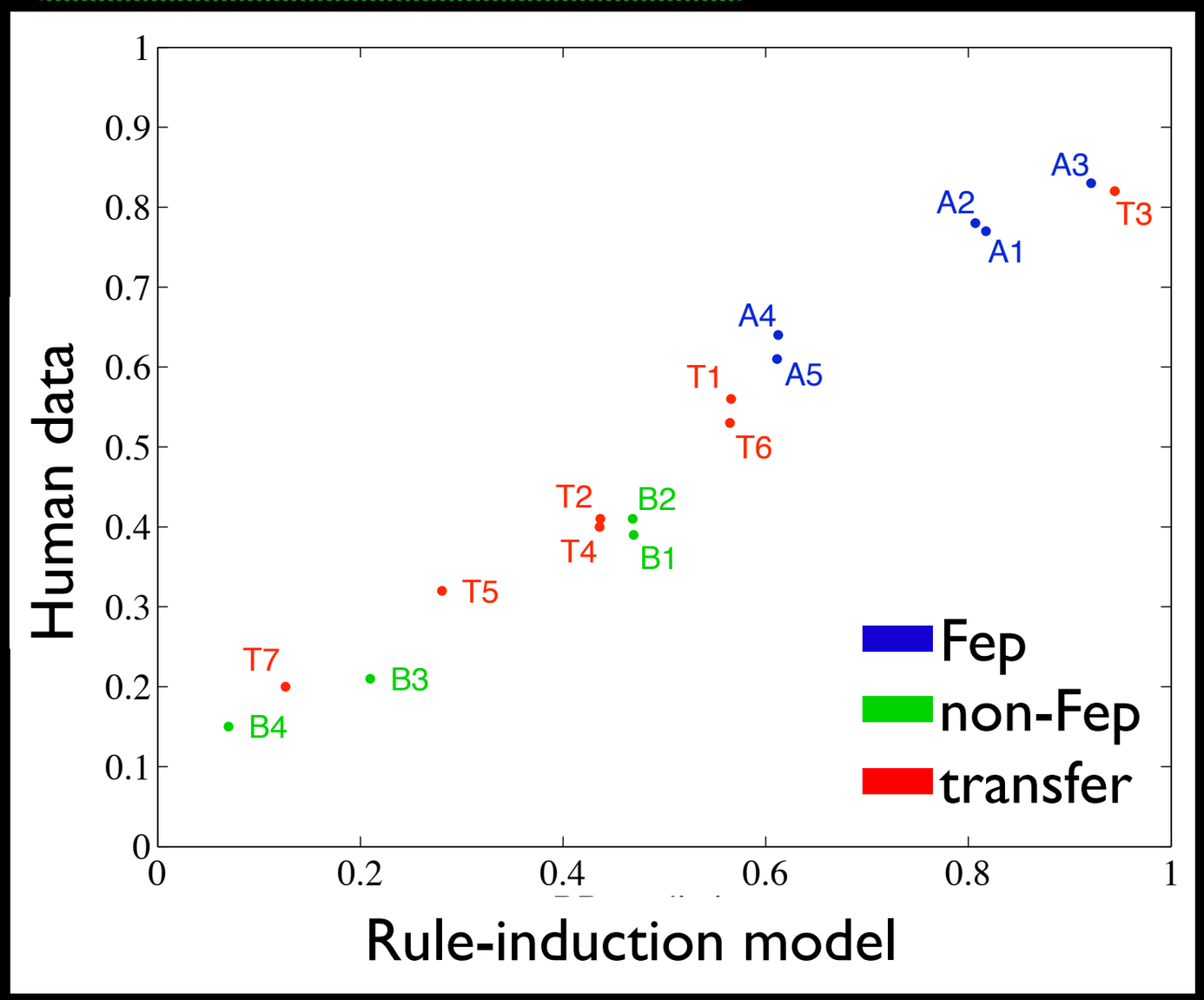


Observation noise:

```
(define noisy
 (λ (bit) (if (flip b) bit (not bit))))
```

Categorization





Goodman, et al.
(2007, 2008a, 2008b)

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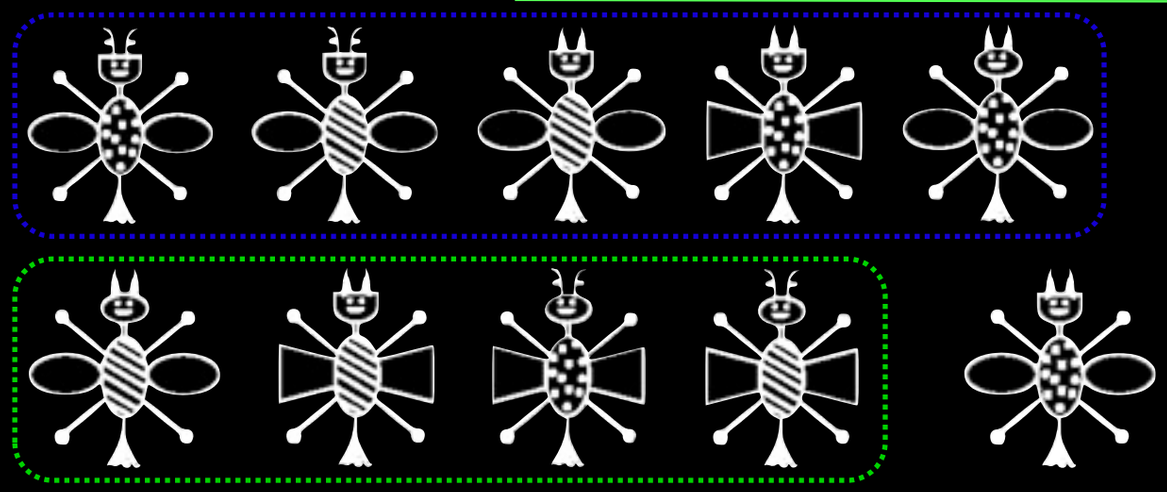

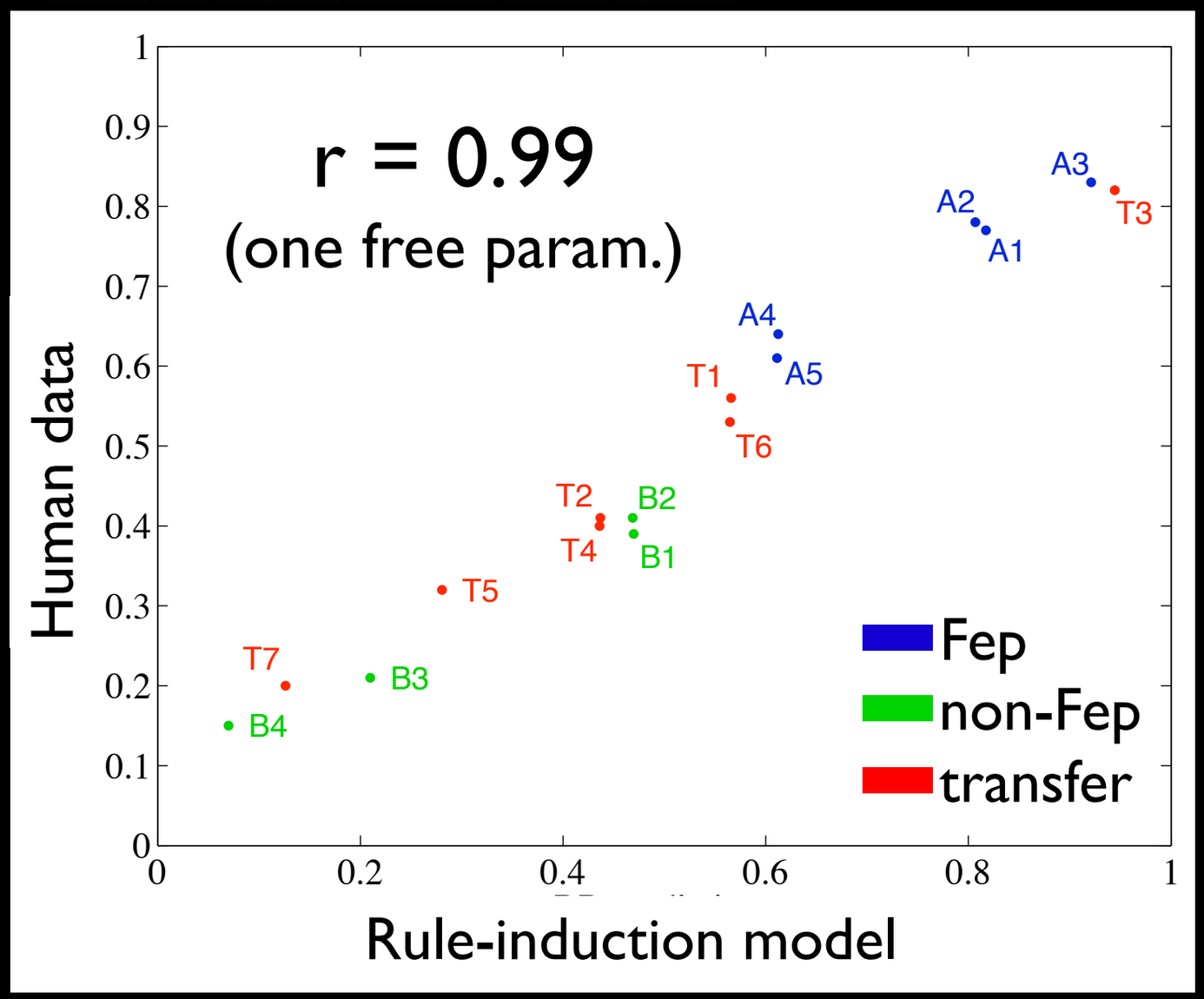

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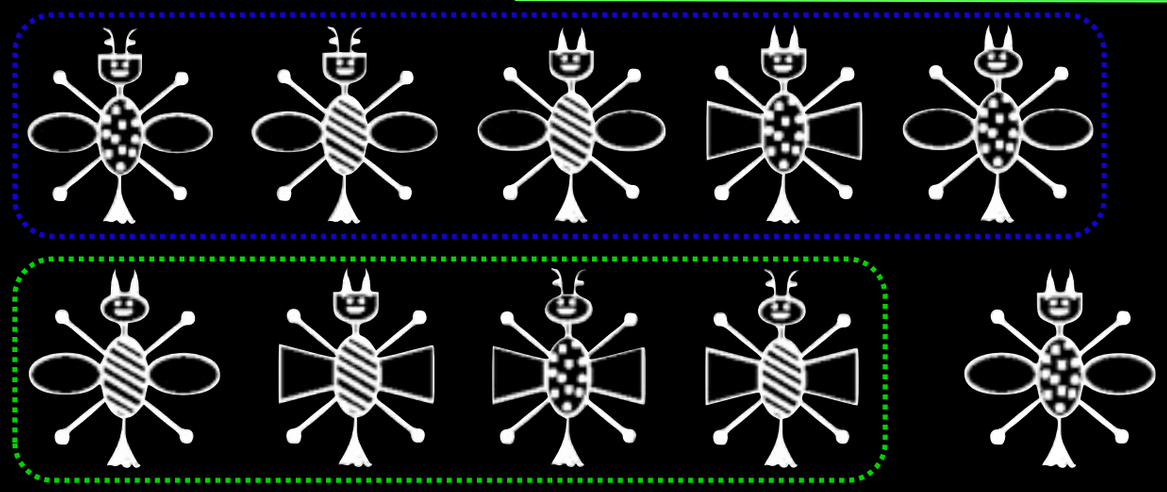
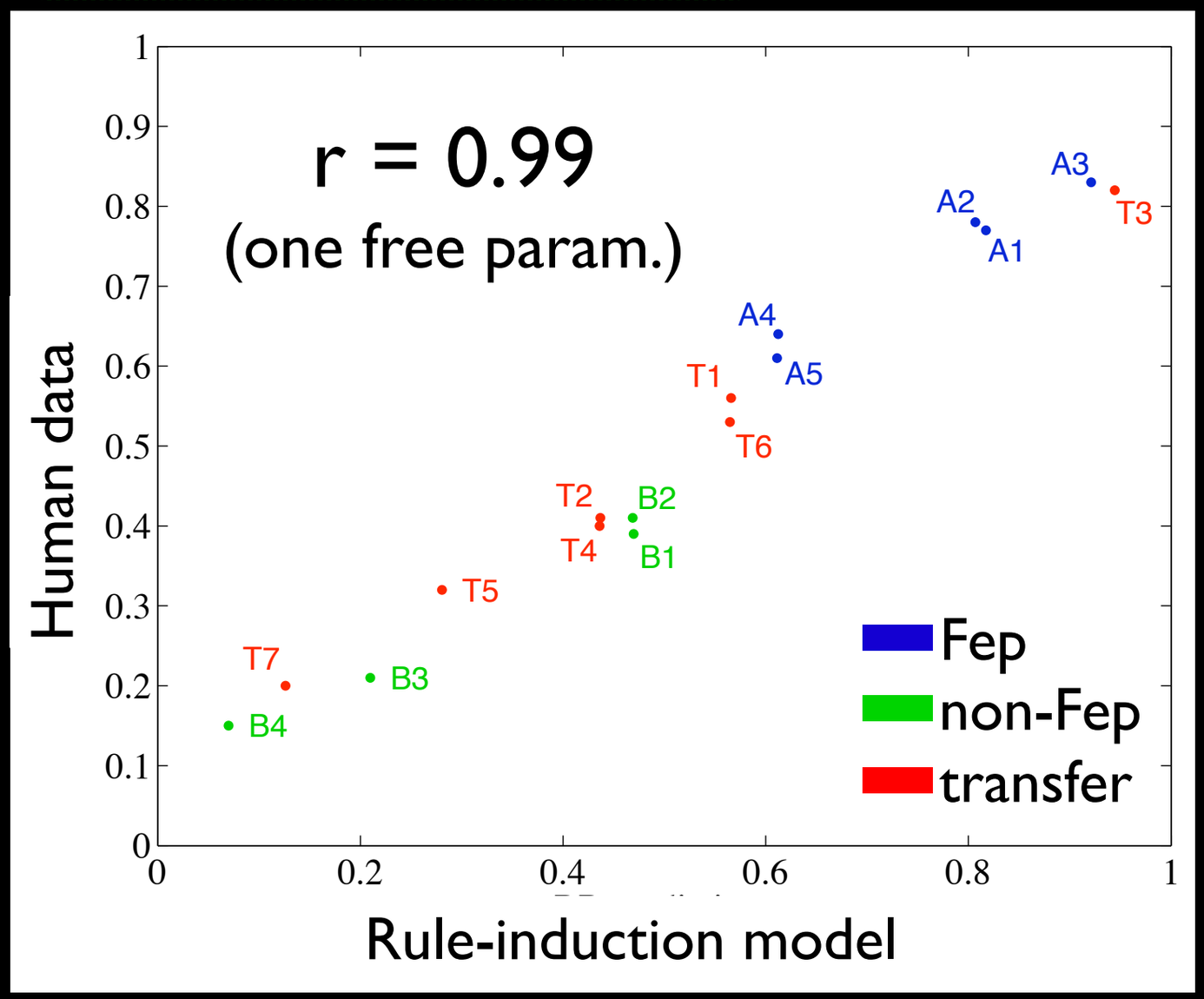
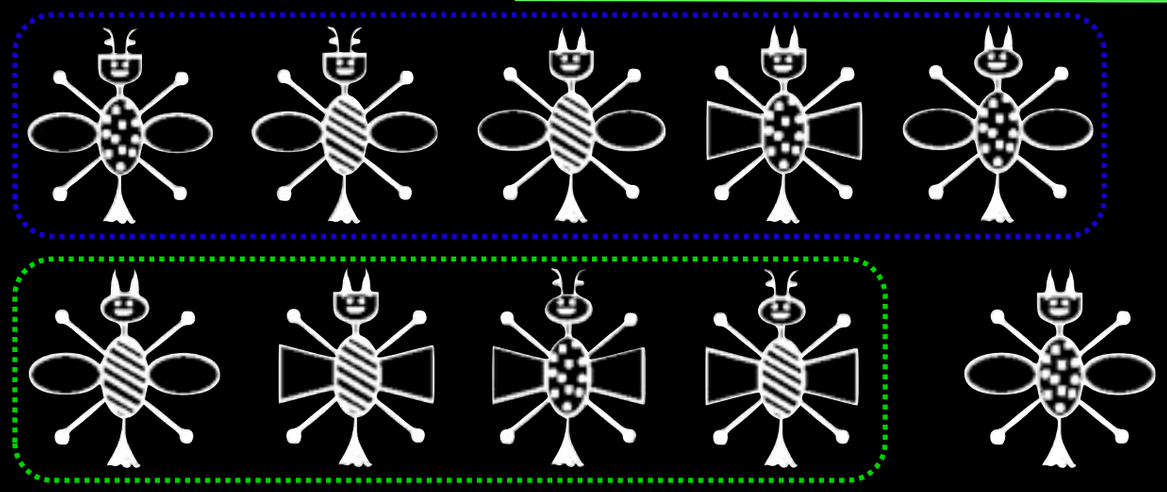


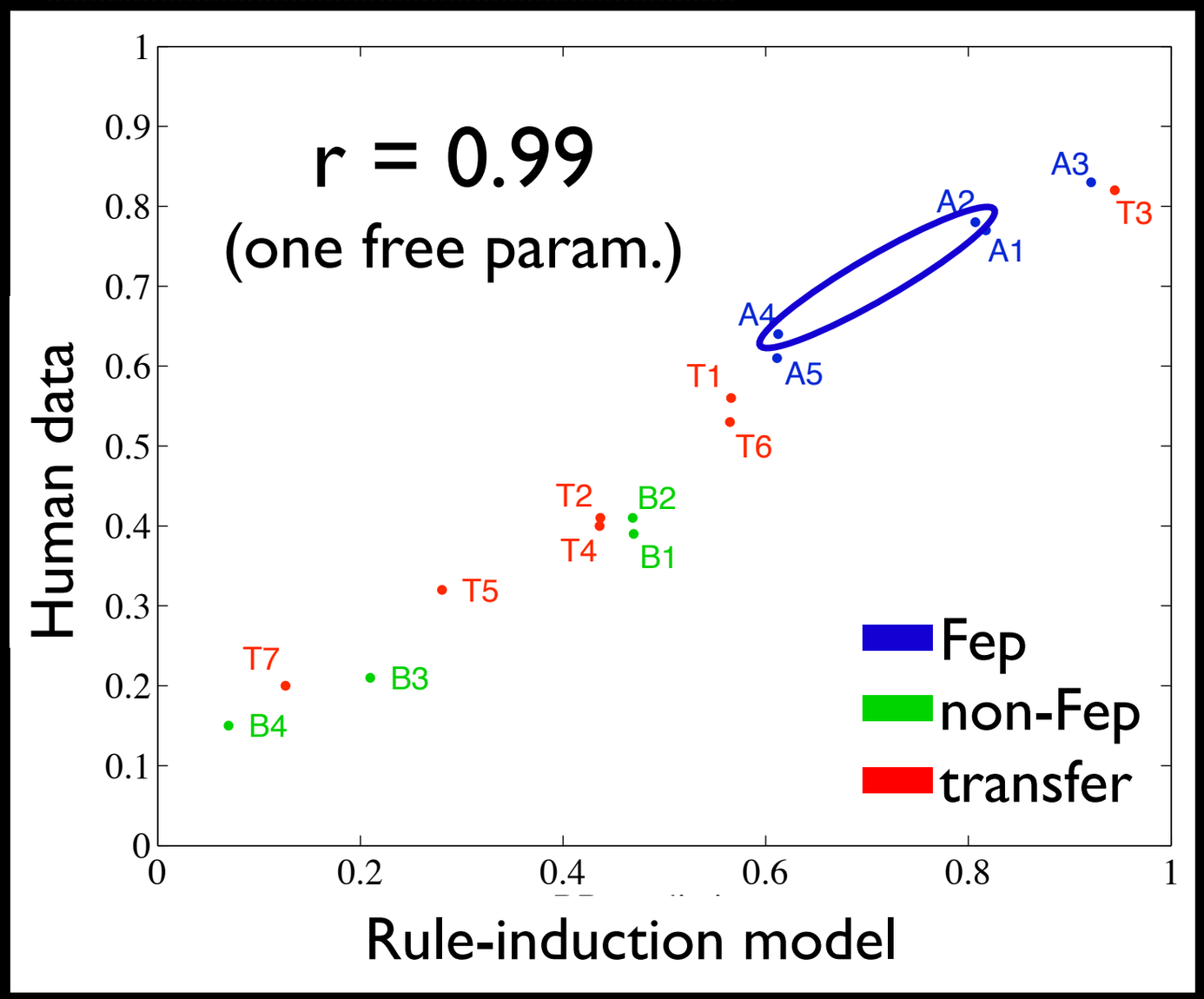


- Graded judgments

Goodman, et al.
(2007, 2008a, 2008b)

Categorization

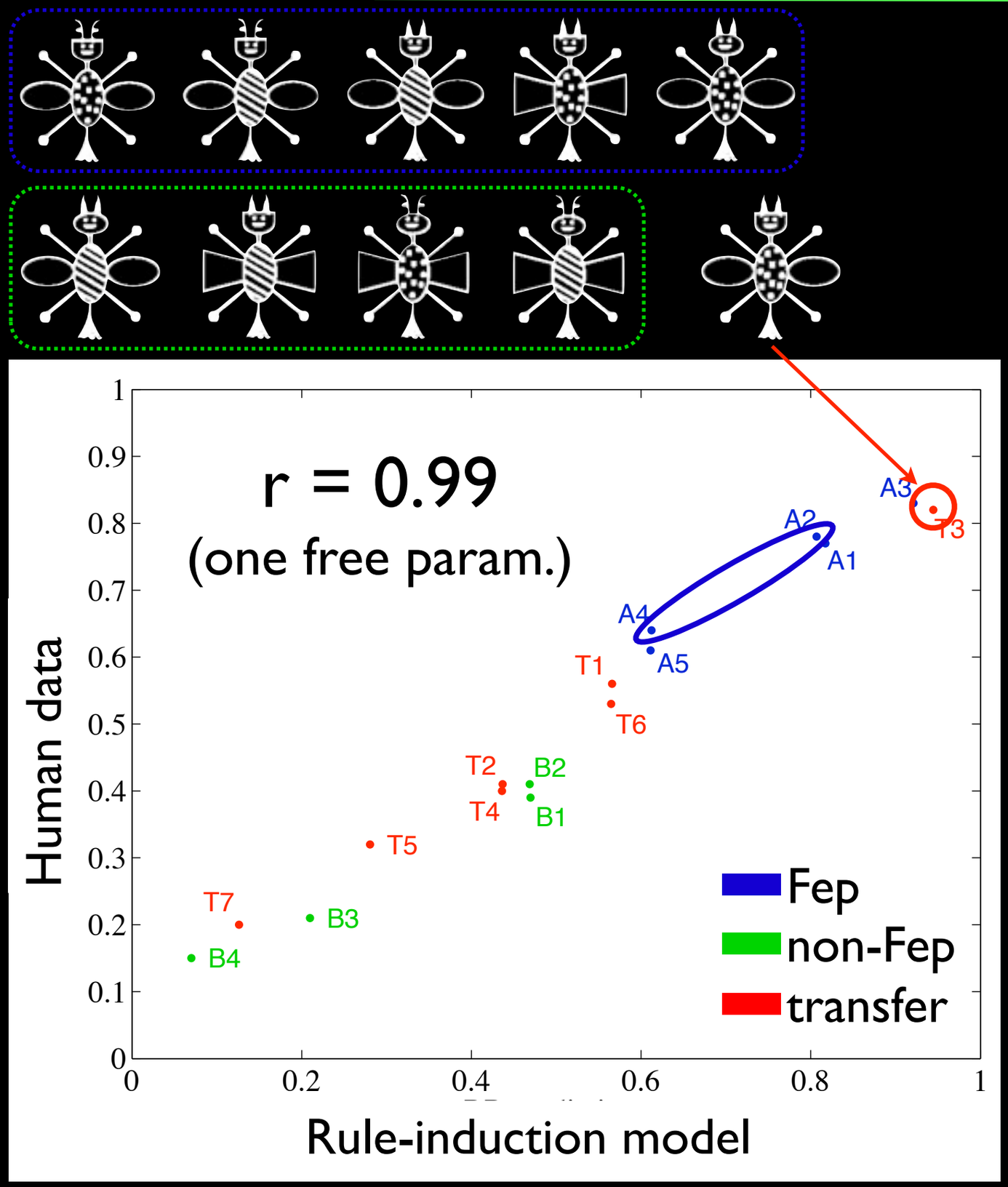




- Graded judgments
- Typicality

Goodman, et al.
(2007, 2008a, 2008b)

Categorization



- Graded judgments
- Typicality
- Prototype enhancement

Goodman, et al. (2007, 2008a, 2008b)

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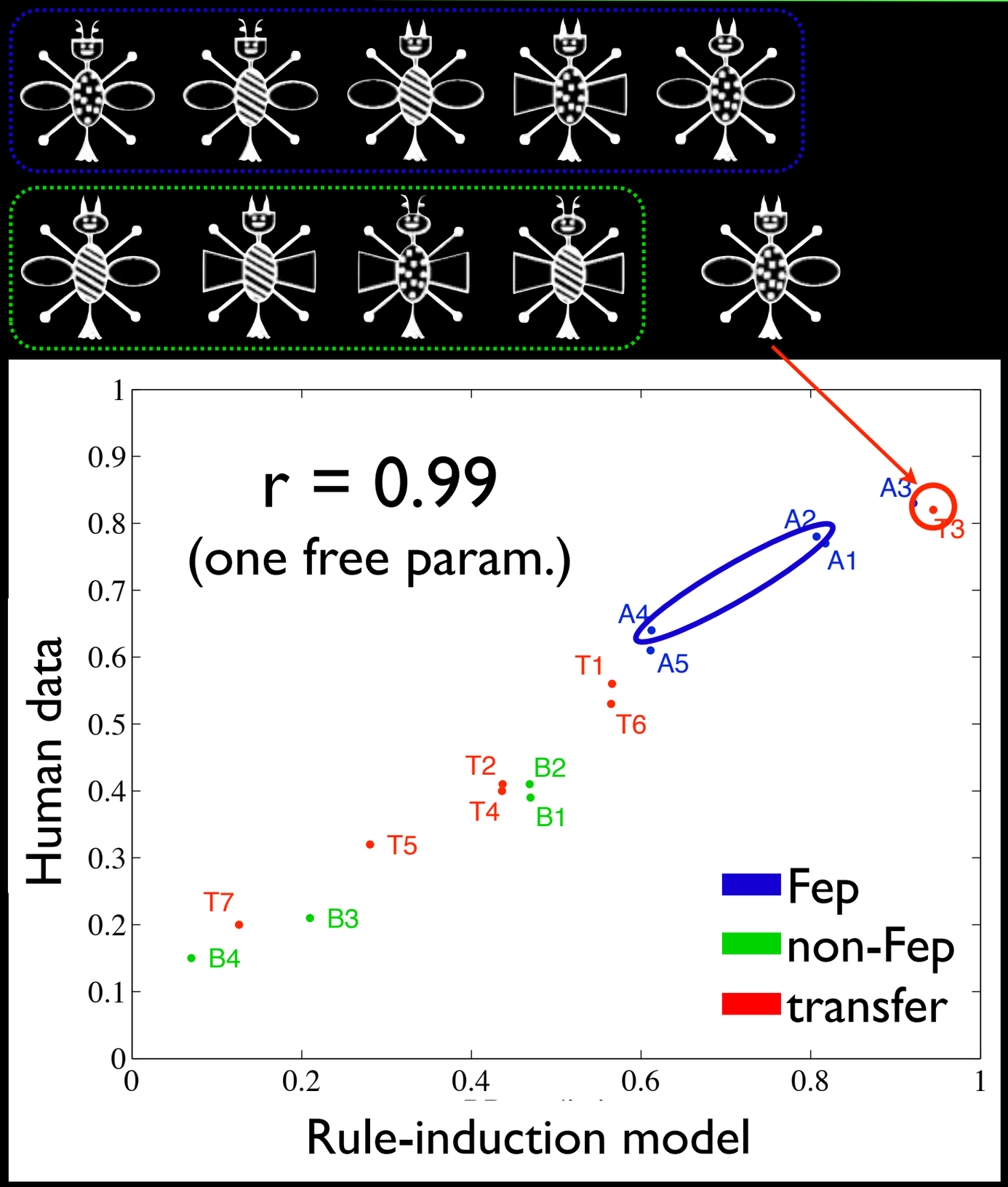


- Graded judgments
- Typicality
- Prototype enhancement
- Selective attention

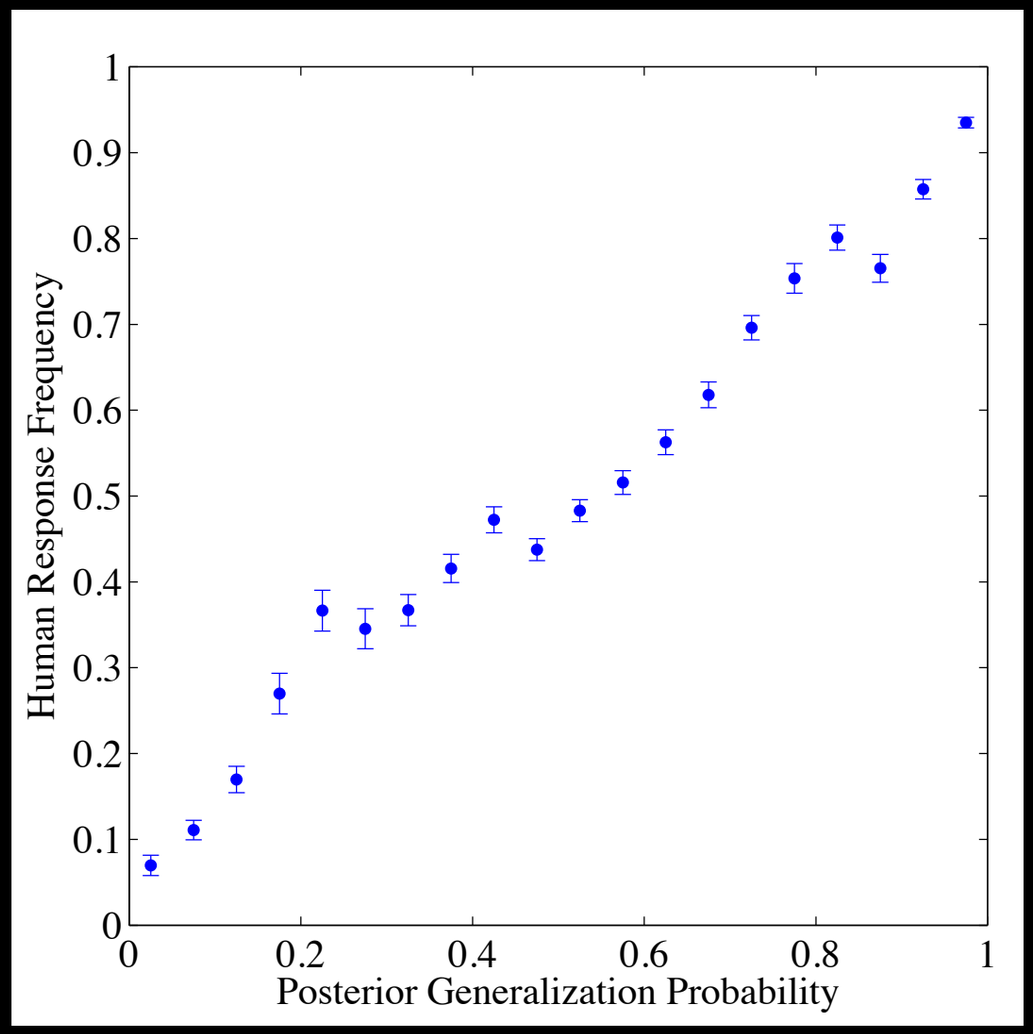
Goodman, et al. (2007, 2008a, 2008b)

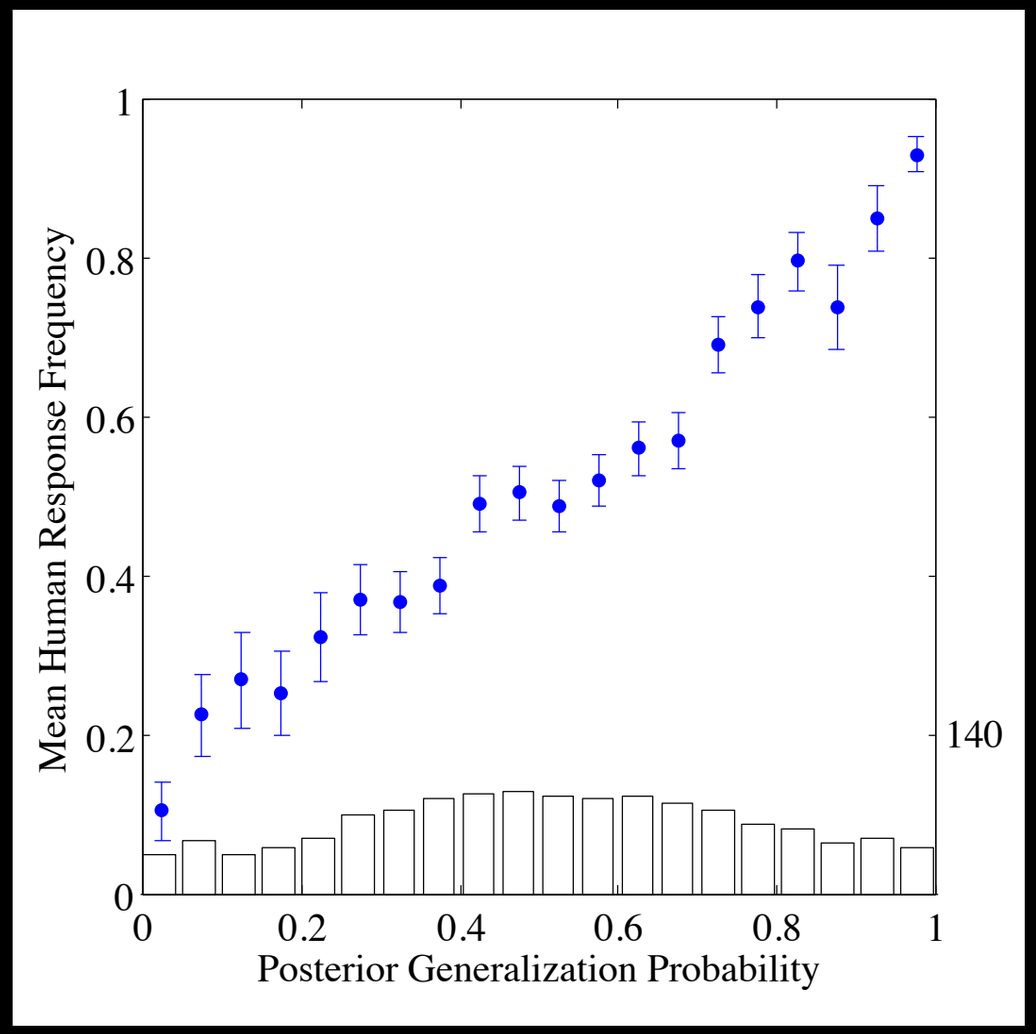
Evaluating languages

- Induction to the language generated by the DNF grammar explains important phenomena (and fits relevant data).
- But is this the *right* LoT?
  - Test on wider data set?
  - Compare to other propositional languages?

Broader test

- 7 Boolean features.
- 43 randomly generated concepts (3-6 pos. + 2 neg. exs)
- 128 judgements (~122 transfer questions)





Goodman, et al (2008)

Evaluating languages

- High-throughput MTurk experiment.
  - 108 concepts,
    - Boolean (*circle or red*)
    - Context-dependent ("Determiners")
      (*unique largest , exists another with same shape*)
  - 2 orders per concept,
  - 1596 participants.

Piantadosi, Goodman, Tenenbaum (in prep)

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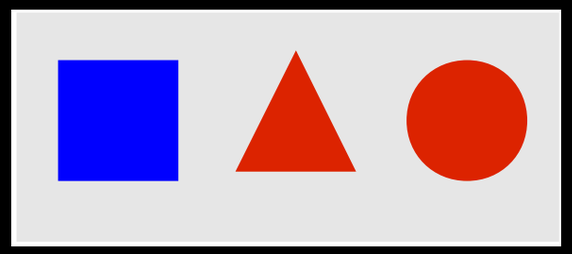

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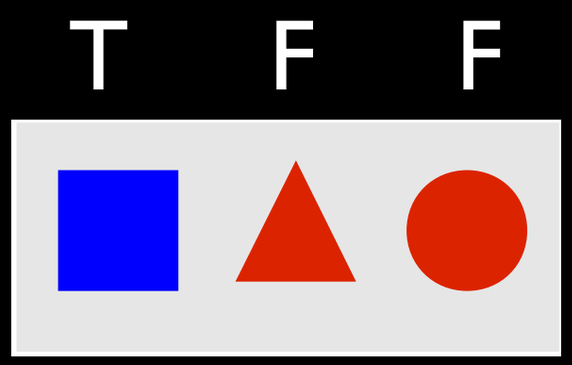

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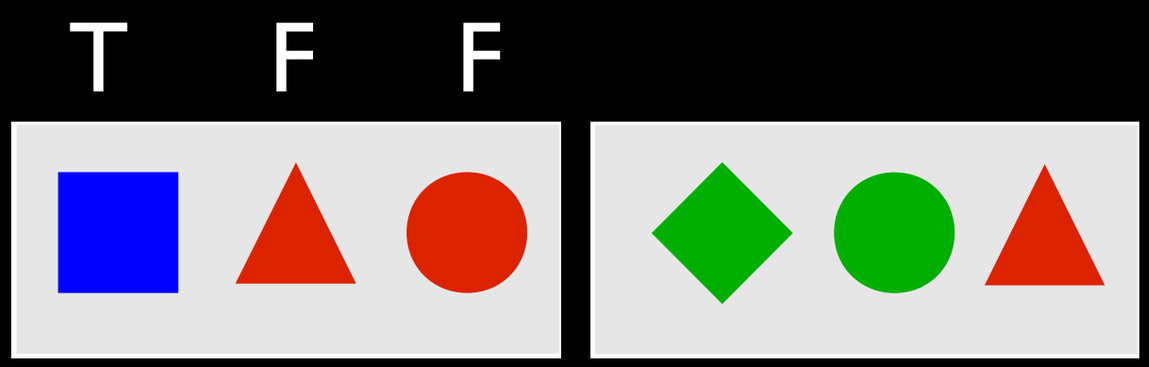

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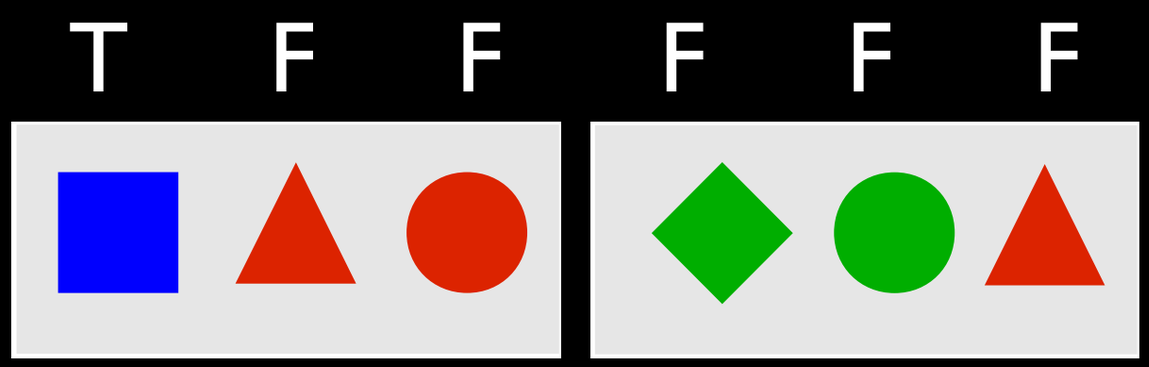

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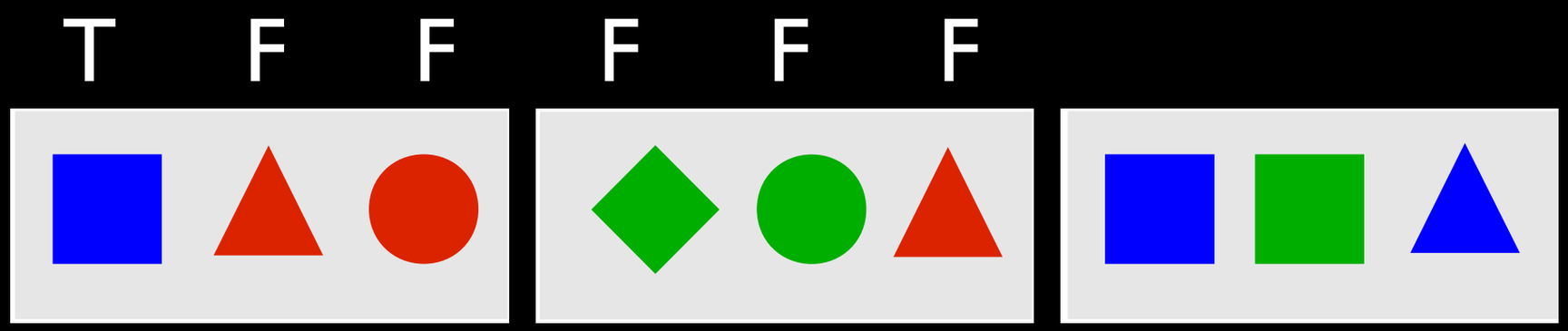

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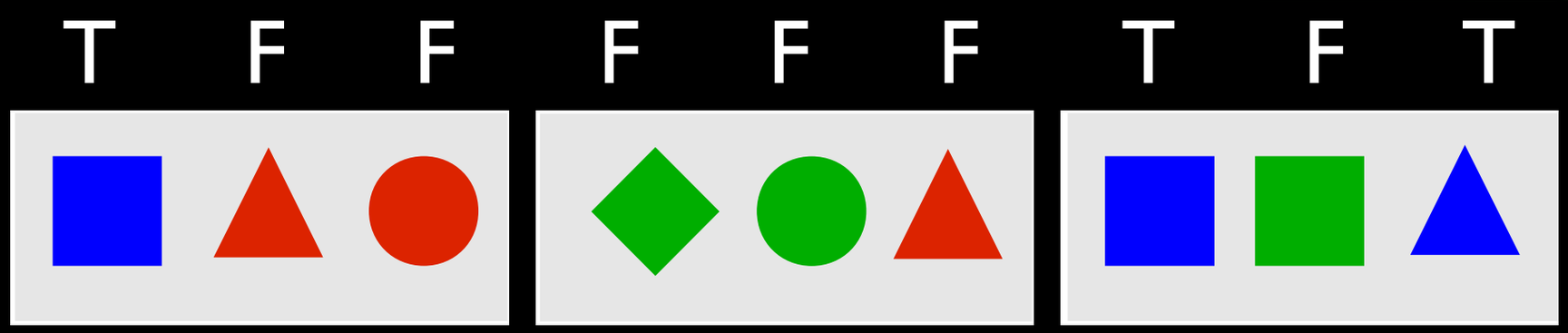

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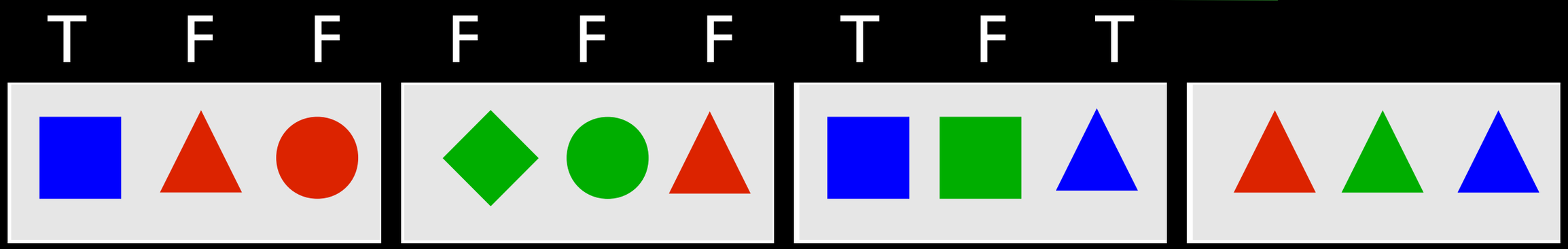

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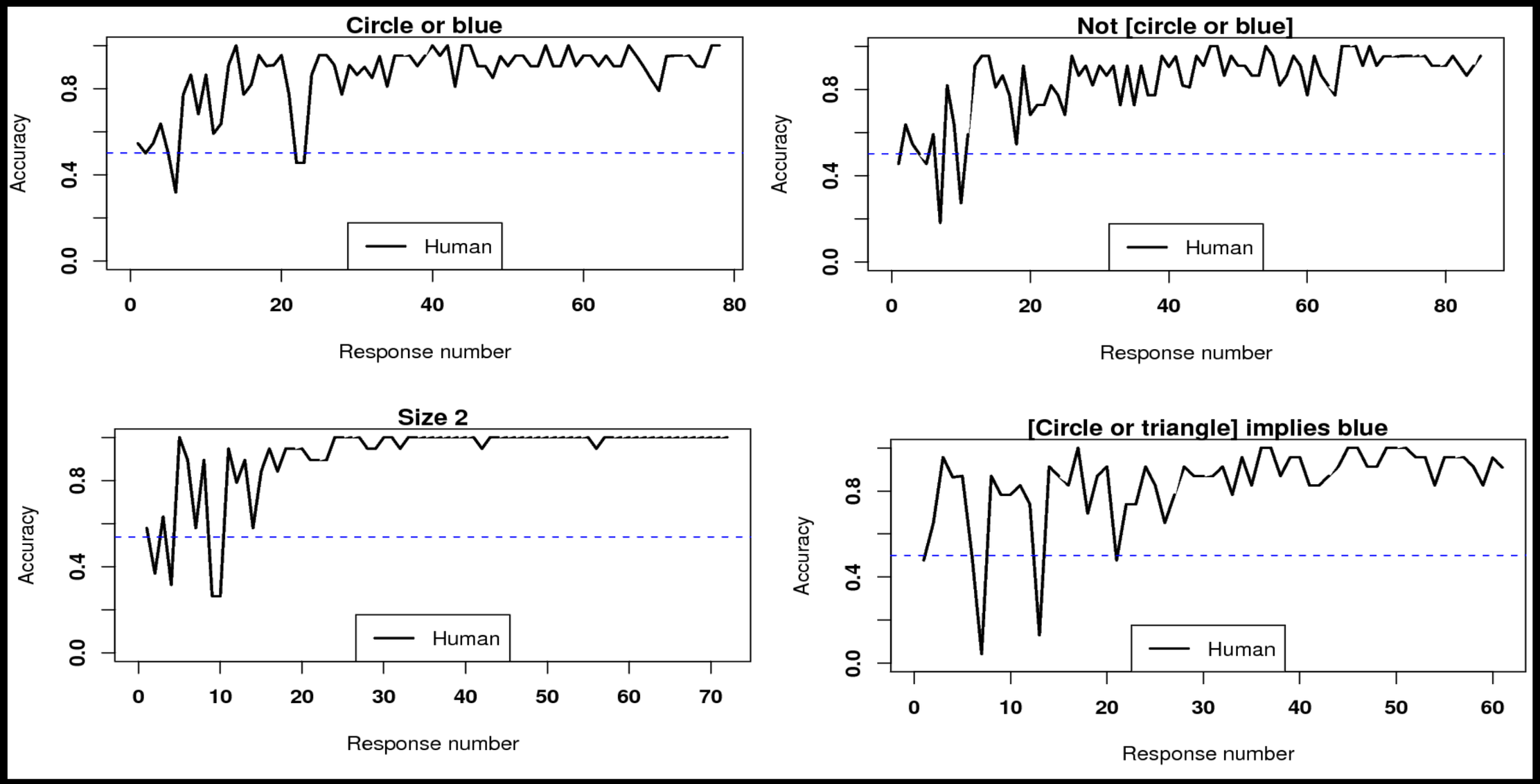
Boolean concepts



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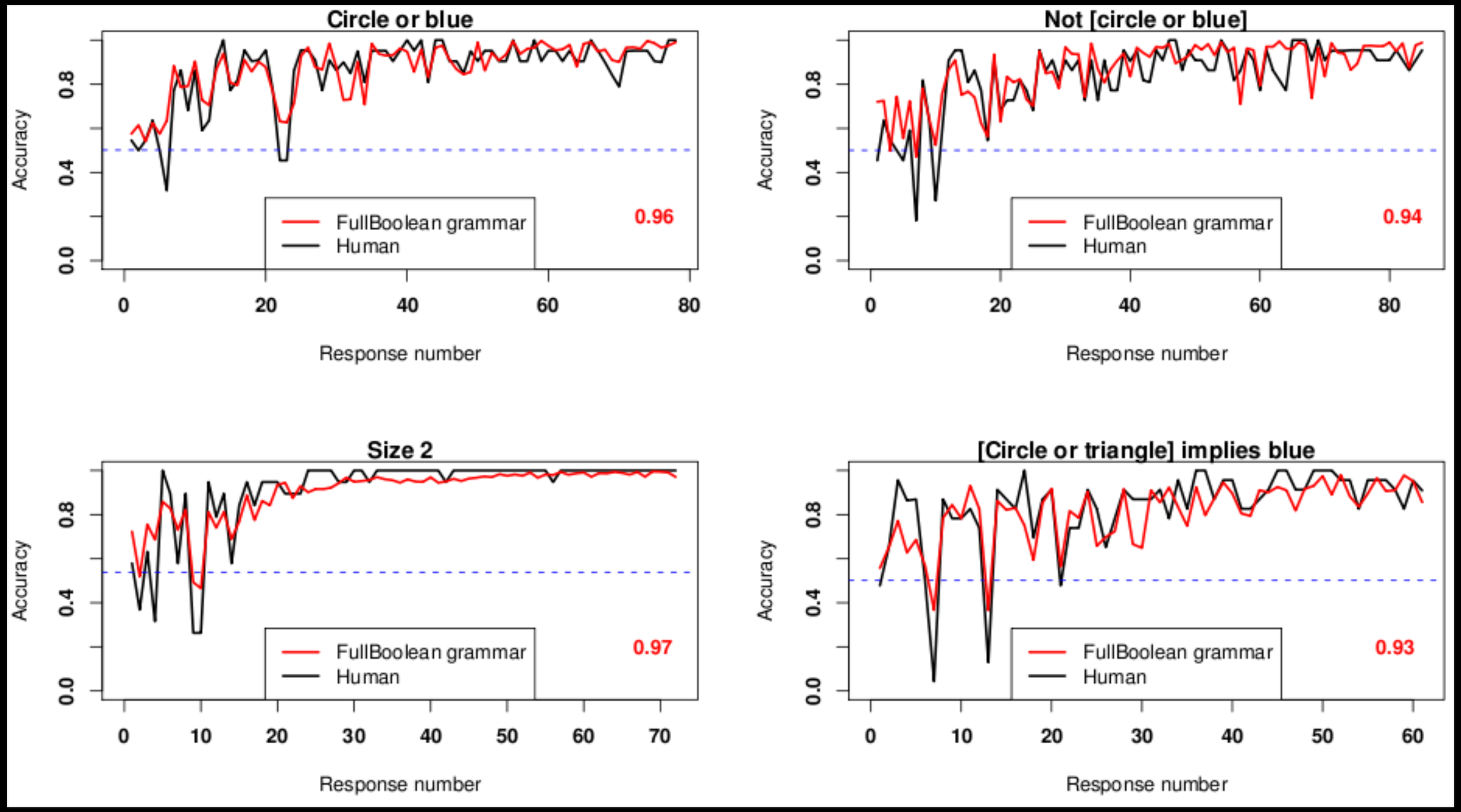
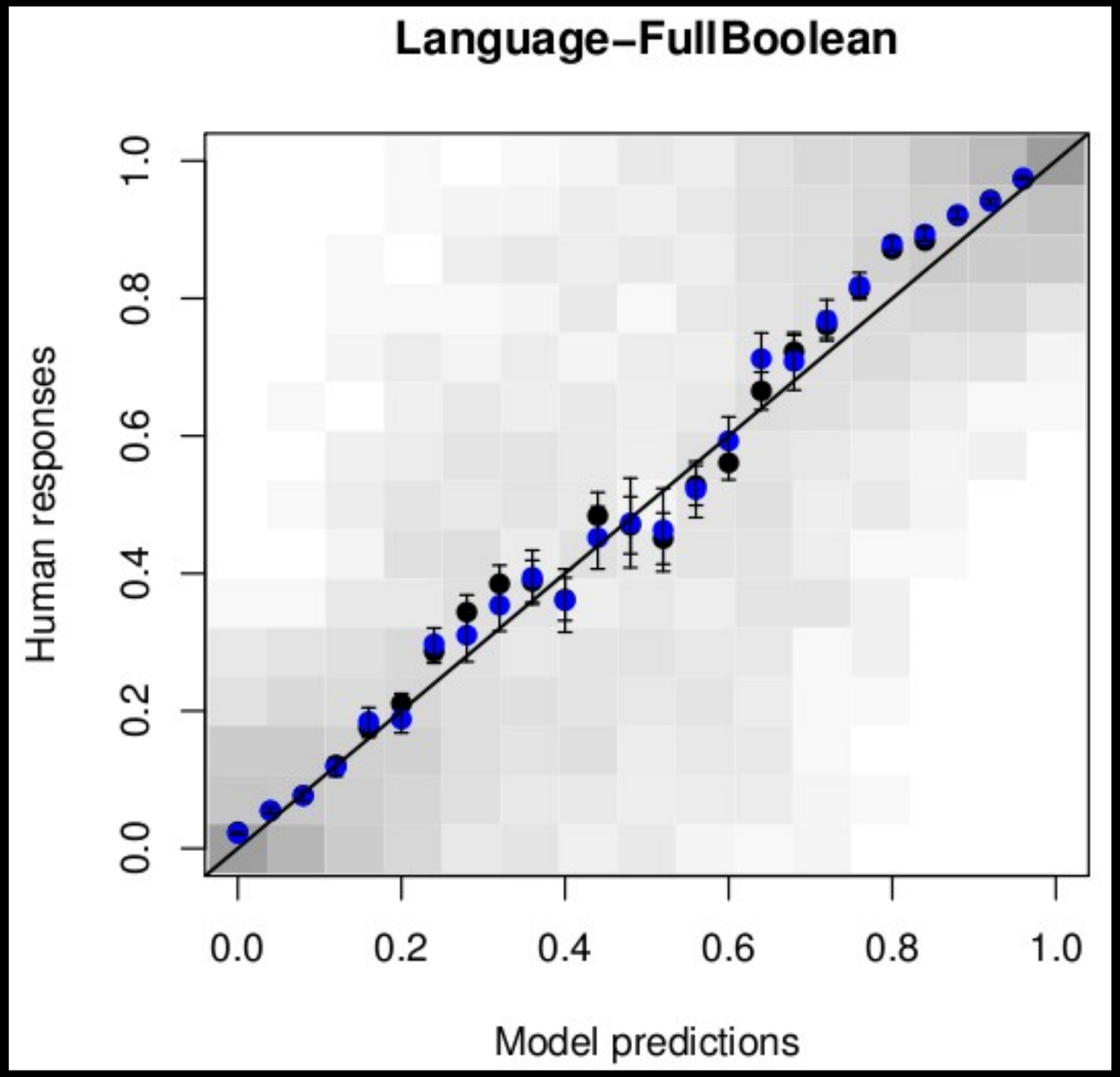


Boolean concepts

Best model performance on Boolean concepts:



Comparing languages

- DNF
  disjunctions of conjunctions

```
(λ (x) (or (and (red? x) (circle? x))
(and (red? x) (triangle? x))))
```

- Horn clauses
  conjunctions of implications

```
(λ (x) (and (implies (not (red? x)) false)
(implies (not (triangle? x)) (circle? x))))
```

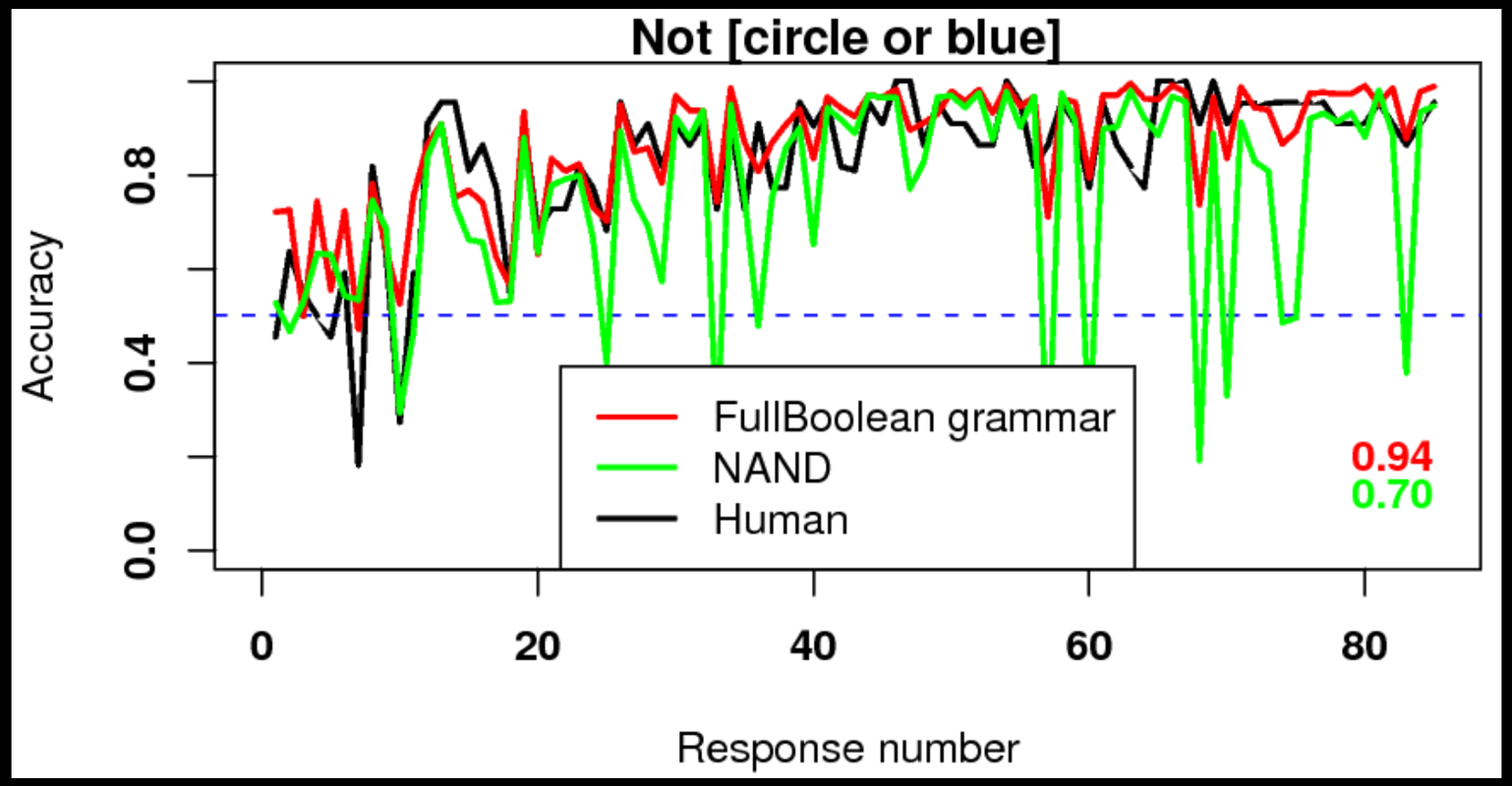
- Full boolean
  any combinations of AND, OR, NOT, IF, IFF

```
(λ (x) (and (red? x) (or (circle? x)
(triangle? x))))
```

- Nand
  combinations of NAND

```
(λ (x) (nand false (nand (red? x)
(nand (nand false (circle? x)) (nand
false (triangle? x)))))
```

Comparing languages

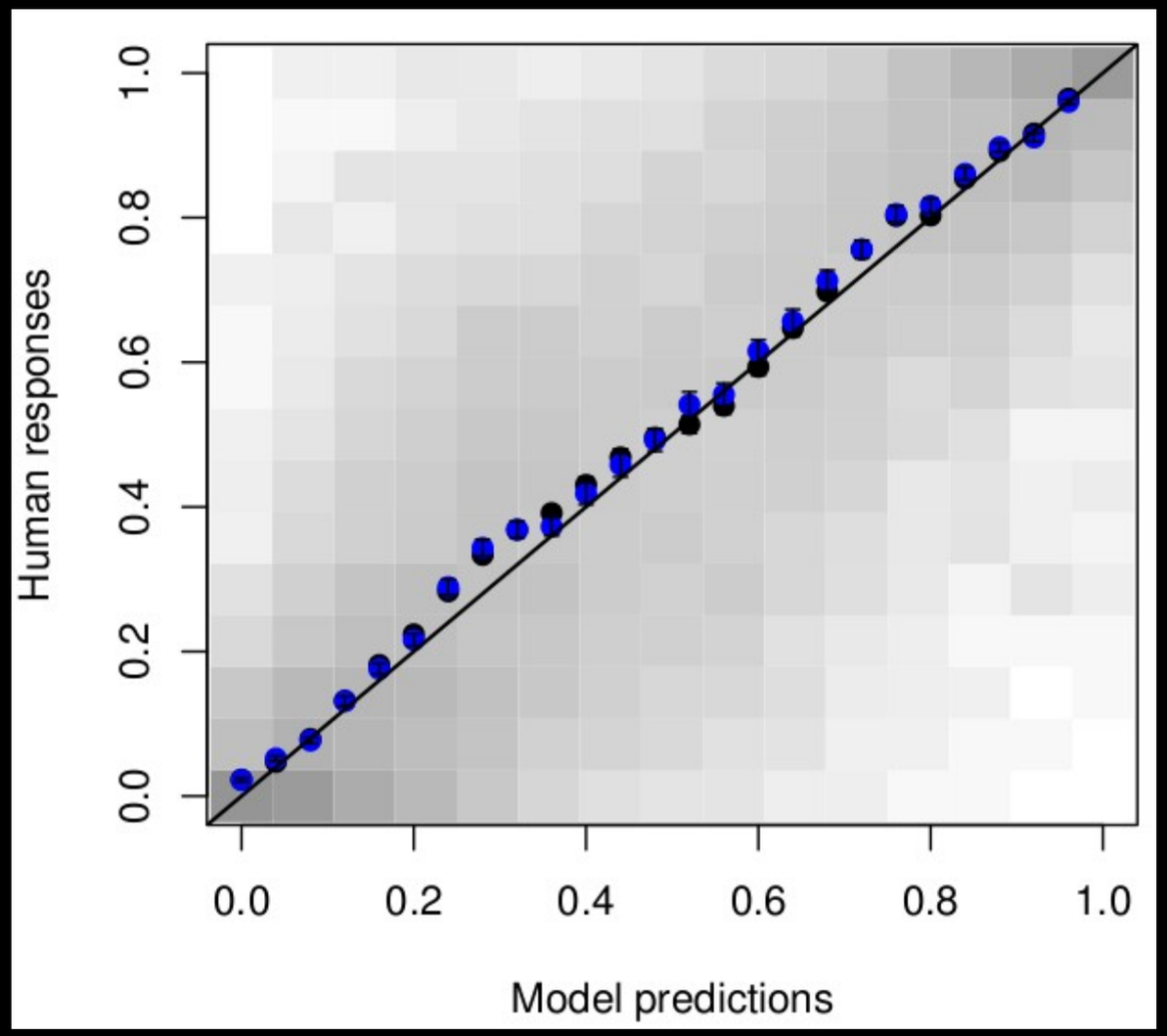


Grammar	H.O. LL
FULLBOOLEAN	-16315.27
CNF	-16333.59
DNF	-16368.31
BICONDITIONAL	-16385.01
IMPLIES	-16442.40
HORNCLAUSE	-16487.25
SIMPLEBOOLEAN	-16490.51
NAND	-16902.68
NOR	-16917.49
UNIFORM	-19482.72
EXEMPLAR	-23645.13
ONLYFEATURES	-31662.08
RESPONSE-BIASED	-37906.77

- Fit hyper-parameters (dirichlet on each NT) for each language.
- Evaluated against held out data.

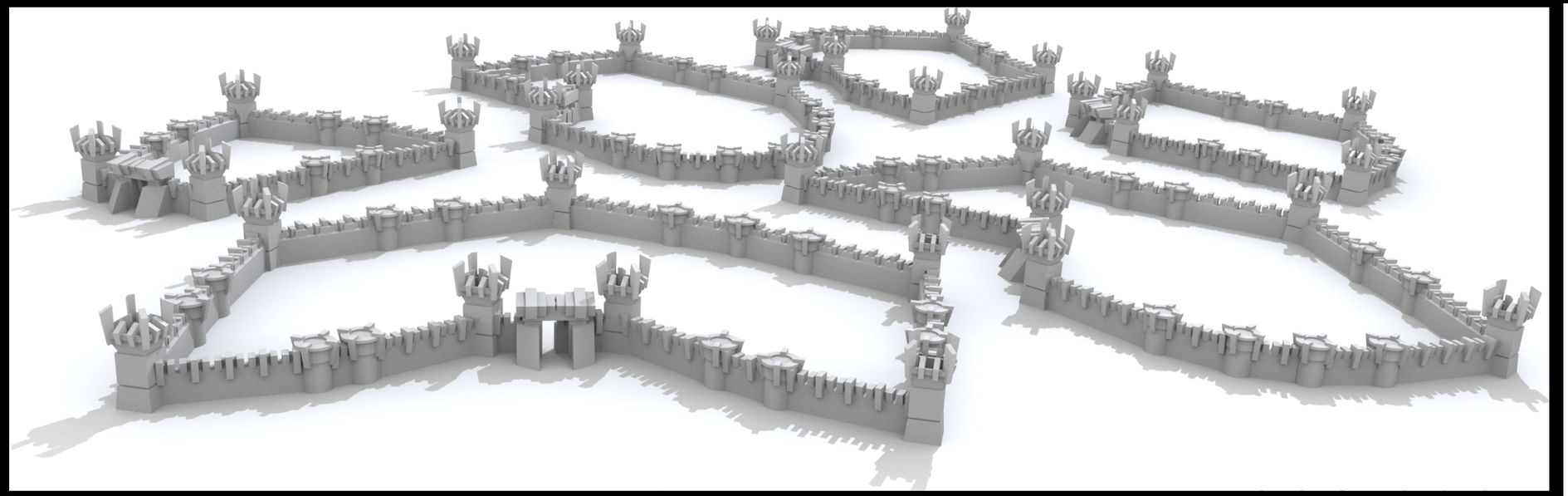
Non-Boolean concepts

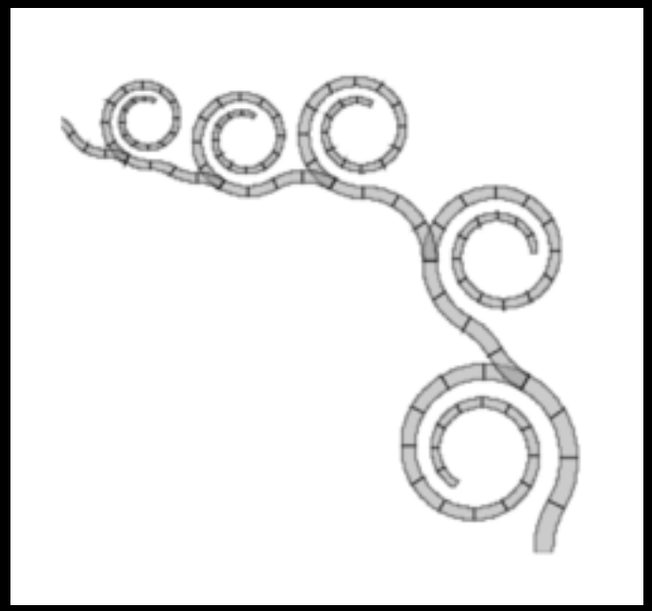
- Best language when context-dependent concepts included is full boolean **plus quantifiers**.

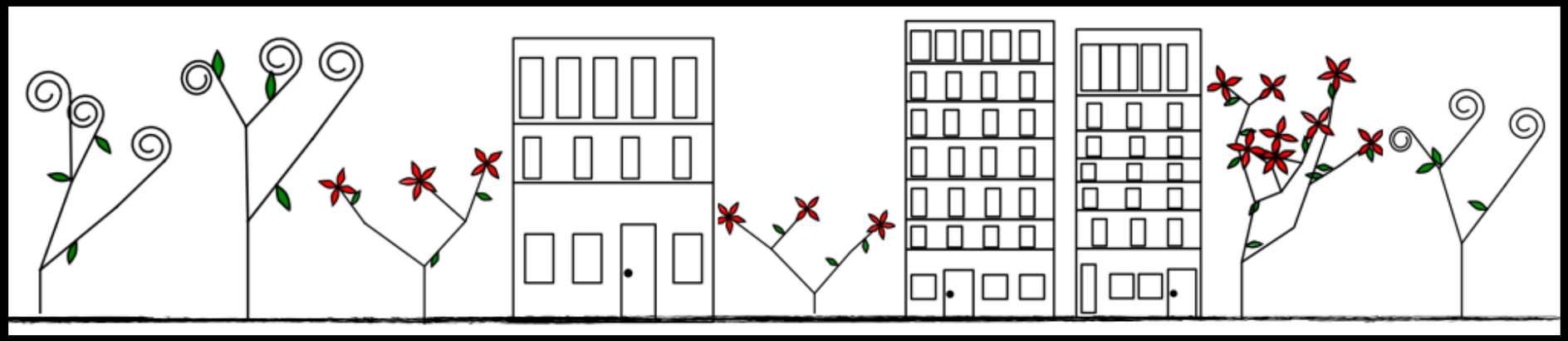


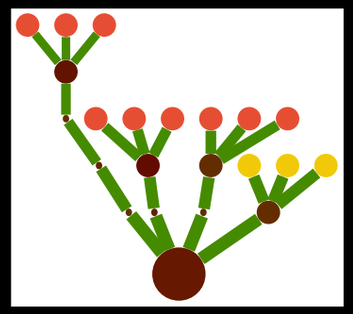
FOL	One-Or-Fewer	Small-Cardinalities	2nd-Ord.-Quan.	H.O. LL
✓	·	·	·	-79279.95
✓	✓	·	·	-79560.90
·	✓	·	·	-79642.46
·	✓	✓	·	-79972.75
✓	✓	·	✓	-80198.75
✓	·	·	✓	-80267.46
✓	·	✓	·	-80285.38
·	✓	·	✓	-80300.00
·	·	✓	·	-80614.35
✓	✓	✓	·	-80942.77
✓	✓	✓	✓	-81138.27
·	✓	✓	✓	-81289.85
✓	·	✓	✓	-81596.68
·	·	✓	✓	-81651.36
FULLBOOLEAN				-81773.43
BICONDITIONAL				-81967.68
SIMPLEBOOLEAN				-82144.71
·	·	·	·	-82219.08
CNF				-82685.21
DNF				-82752.82
·	·	·	✓	-82853.59

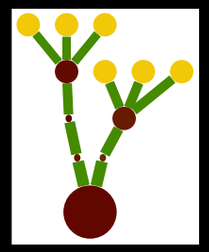
Generative kinds

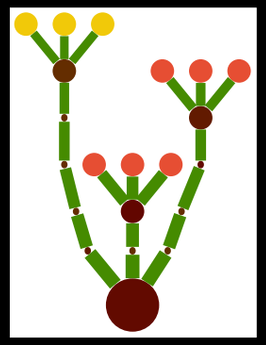


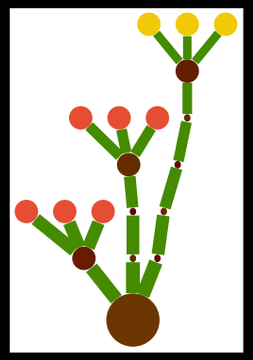


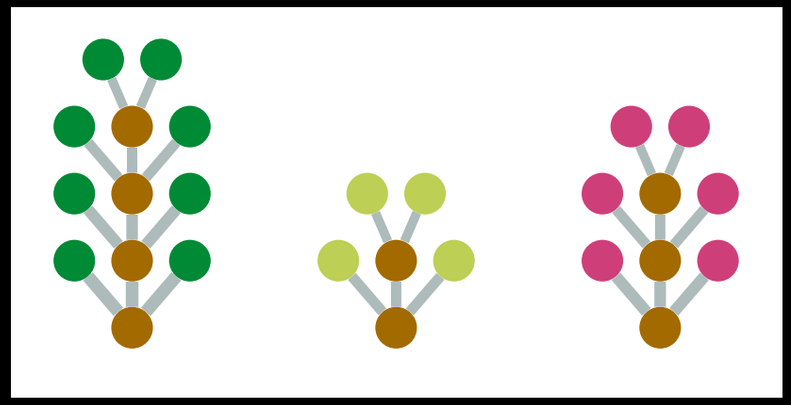








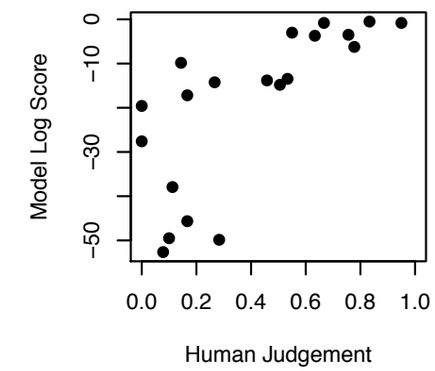




Learning generative kinds

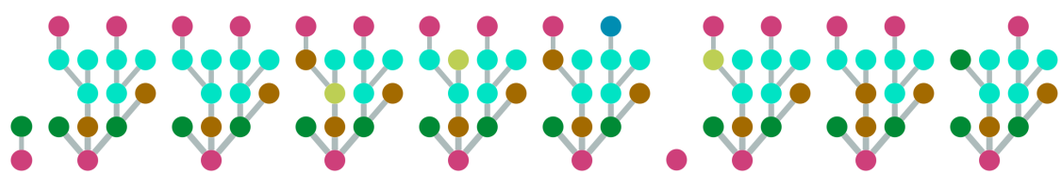
Simple prototype

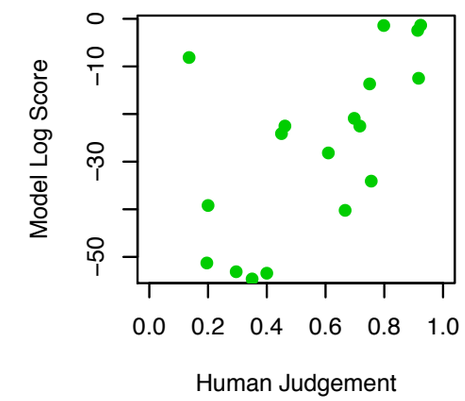
```
(lambda ()
  (node 'a
        (node 'b (node 'c (node 'c)) (node 'b))
        (node 'd (node 'e))))
```



Sub-concepts

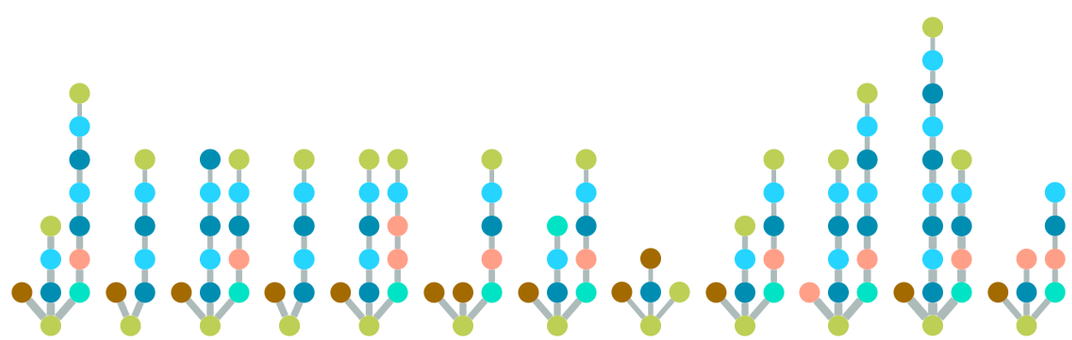
```
(begin
  (define (part) (node 'c (node 'c (node 'a)) (node 'c)))
  (lambda () (node 'a (node 'b) (node 'd (part)) (node 'b (part) (node 'd)))))
```

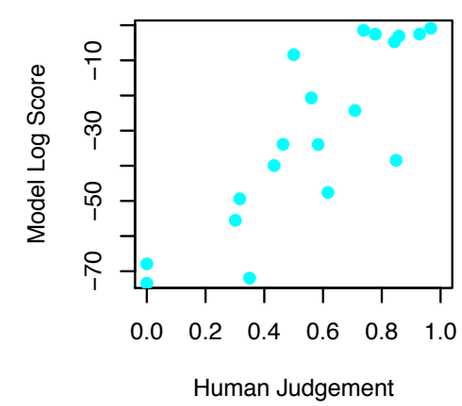




Single recursion

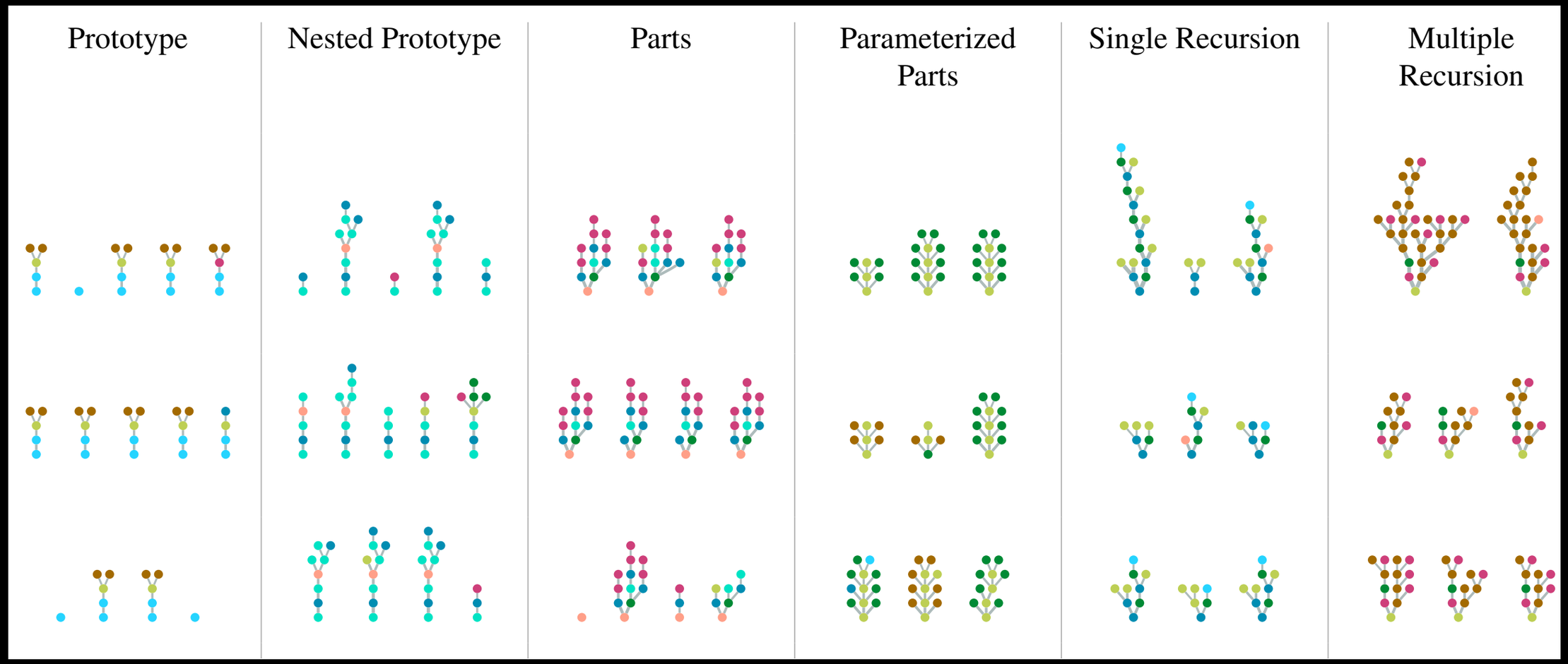
```
(begin
  (define (part) (node 'a (node 'b (if (flip) (node 'c) (part)))))
  (lambda () (node 'c (node 'd) (part) (node 'e (node 'f (part))))))
```





Stuhlmueller, Tenenbaum, Goodman (2010)

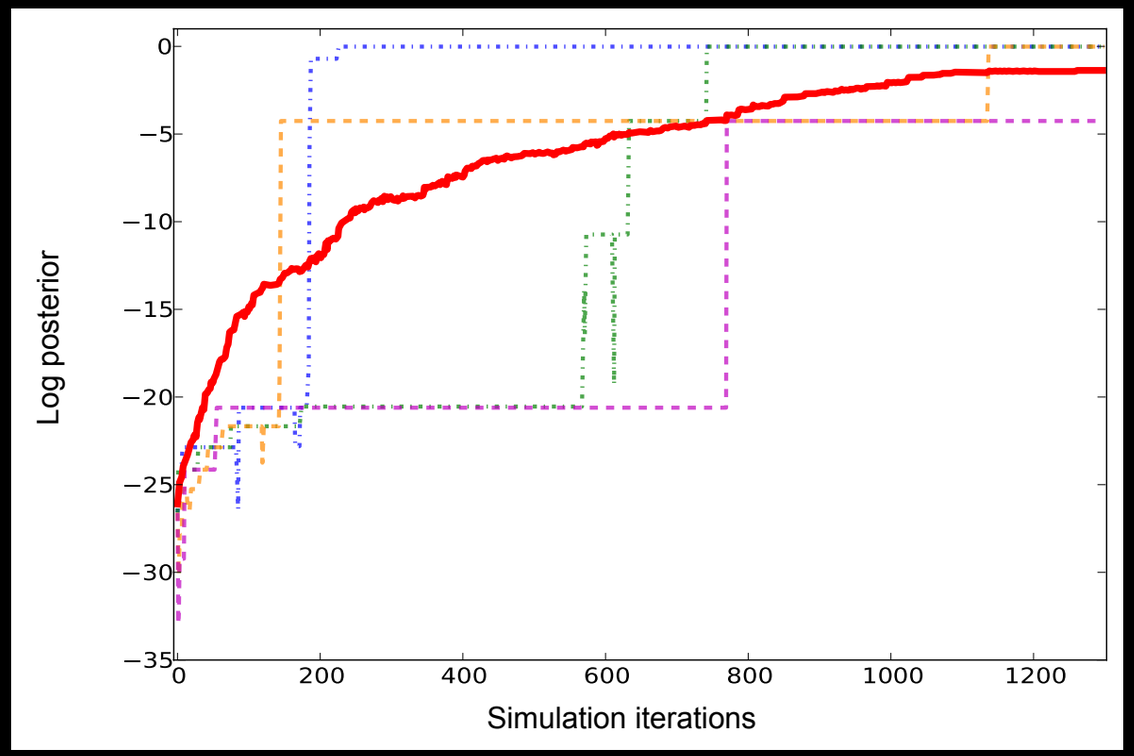
Learning generative kinds



	Set GCM	Transition GCM	Tree GCM	Generative Model
Prototype	0.589	0.751	**0.803**	0.748
Nested Prototype	0.544	0.851	**0.937**	0.904
Parts*	0.320	0.617	0.705	**0.835**
Parameterized Parts	0.298	0.591	0.778	**0.911**
Single Recursion	0.284	0.499	0.637	**0.773**
Multiple Recursion	0.505	0.561	0.451	**0.770**

Algorithms for induction

- What algorithms are capable of learning concepts in a language of thought?
  - All results so far were computed using MCMC based on constituent regeneration (Goodman, et al, 2008).
  - Is this cognitively plausible? Maybe...
  - But this is probably not enough on its own.



(Ullman, Goodman, Tenenbaum, 2010)

Program induction

- Program induction is especially hard. How could it be done?
- Idea: anti-unification + argument compression (+ search/MCMC).

```
(/ (+ 1 (* 2 3))
   (+ 1 (* 5 3)))
```

"inverse inlining" →

```
(define (F x) (+ 1 (* x 3)))
(/ (F 2) (F 5))
```

```
(define (F x)
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(/ (F 2) (F 5))
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"de-argument" →

```
(define (F)
 (+ 1 (* (gaussian 3.5 1.0) 3)))
(/ (F) (F))
```

```
(define (F x)
 (+ 1 (* x 3)))
(F (F 1))
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(define (F)
 (if (flip) (+ 1 (* (F) 3)) 1))
(F)
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   (+ 1 (* 5 3)))
```

"inverse inlining" →

```
(define (F x) (+ 1 (* x 3)))
(/ (F 2) (F 5))
```

```
(define (F x)
 (+ 1 (* x 3)))
(/ (F 2) (F 5))
```

"de-argument" →

```
(define (F)
 (+ 1 (* (gaussian 3.5 1.0) 3)))
(/ (F) (F))
```

```
(define (F x)
 (+ 1 (* x 3)))
(F (F 1))
```

"de-argument" →

```
(define (F)
 (if (flip) (+ 1 (* (F) 3)) 1))
(F)
```

Program induction

- Program induction is especially hard. How could it be done?
- Idea: anti-unification + argument compression (+ search/MCMC).

```
(/ (+ 1 (* 2 3))
   (+ 1 (* 5 3)))
```

"inverse inlining" →

```
(define (F x) (+ 1 (* x 3)))
(/ (F 2) (F 5))
```

```
(define (F x)
 (+ 1 (* x 3)))
(/ (F 2) (F 5))
```

"de-argument" →

```
(define (F)
 (+ 1 (* (gaussian 3.5 1.0) 3)))
(/ (F) (F))
```

```
(define (F x)
 (+ 1 (* x 3)))
(F (F 1))
```

"de-argument" →

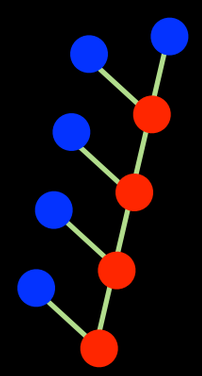
```
(define (F)
 (if (flip) (+ 1 (* (F) 3)) 1))
(F)
```

Program induction

- *Bayesian program merging* algorithm:
  (Cf. Stolcke & Omohundro, 1994)
  - Initial state is exemplar program (mixture of data).
  - Transform programs via anti-unification and argument compression.
  - Beam search* with respect to the posterior score.
    (*Or stochastic search, Monte Carlo, etc.)
    - Likelihood: marginal probability of data given program (computed by lazy particle filter or dynamic progr.).
    - Prior: syntactic complexity.

Hwang, Stuhlmueller, Goodman (2011)

Example



"data incorporation" →

```
(node r (node b)
        (node r (node b)
                (node r (node b)
                        (node r (node b) b)
```

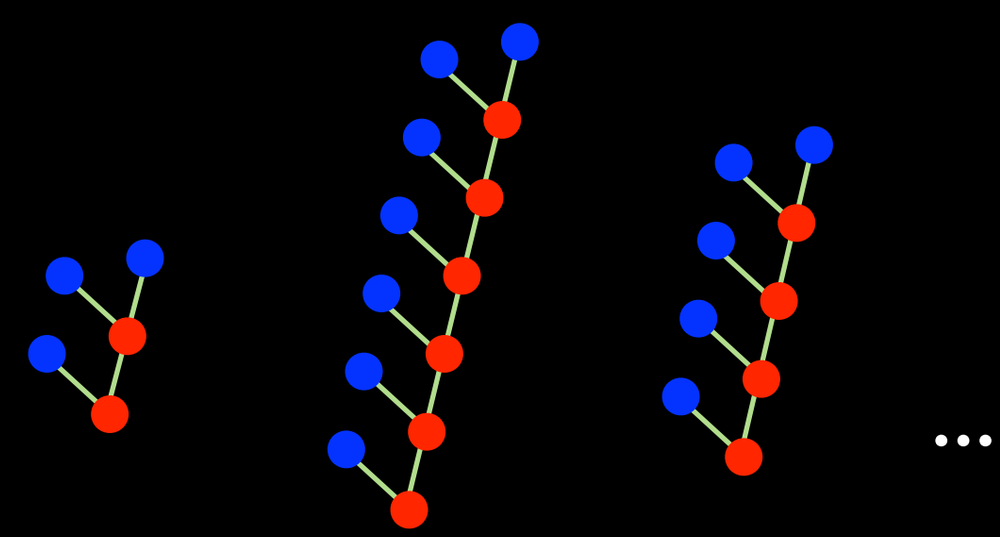
"inverse inline" →

```
(define (F x) (node r (node b) x))
(F (F (F (F b))))
```

"de-argument" →

```
(define (F) (if (flip 0.8)
                (node r (node b) (F))
                b))
(F)
```

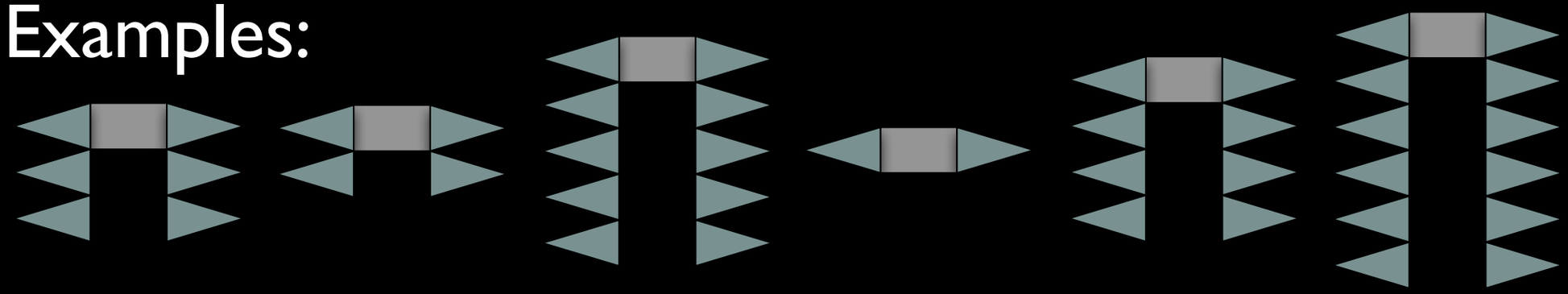
sample ⇢



...

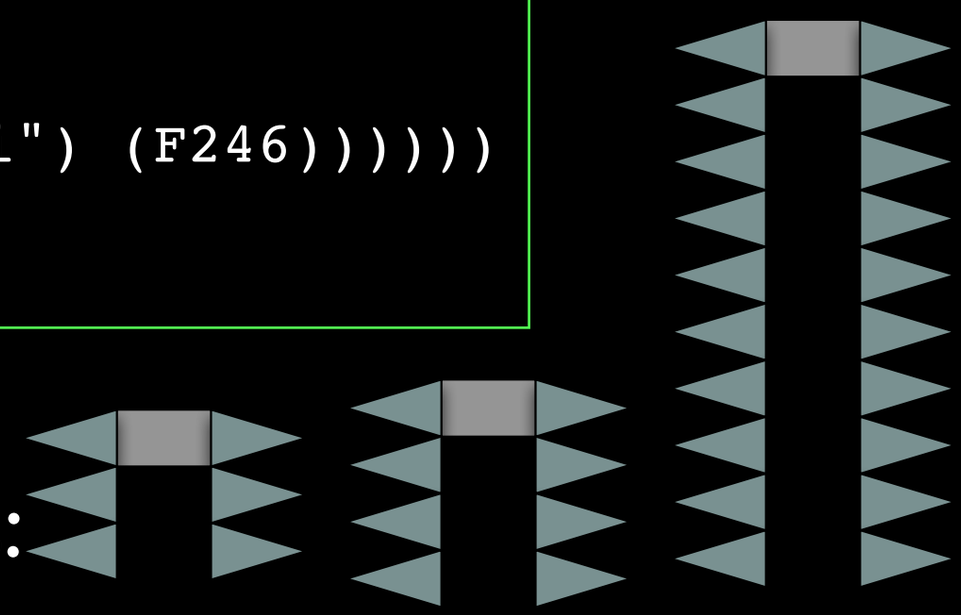
Example

Examples:

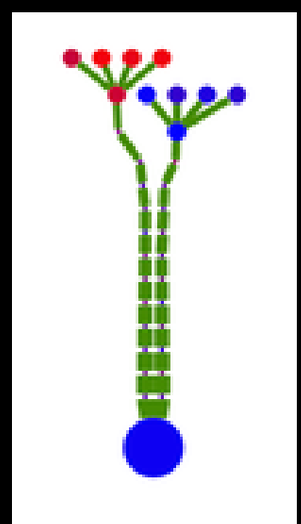


```
(define F248 (lambda () (F247 (F246))))

(define F247 (lambda (V994)
               (elem "rect"
                     (tr "left" (tr "reflect" V994))
                     (tr "right" V994))))

(define F246 (lambda ()
               (elem "tri"
                     (tr "forward"
                         (choose (elem "tri") (F246))))))

(F248)
```

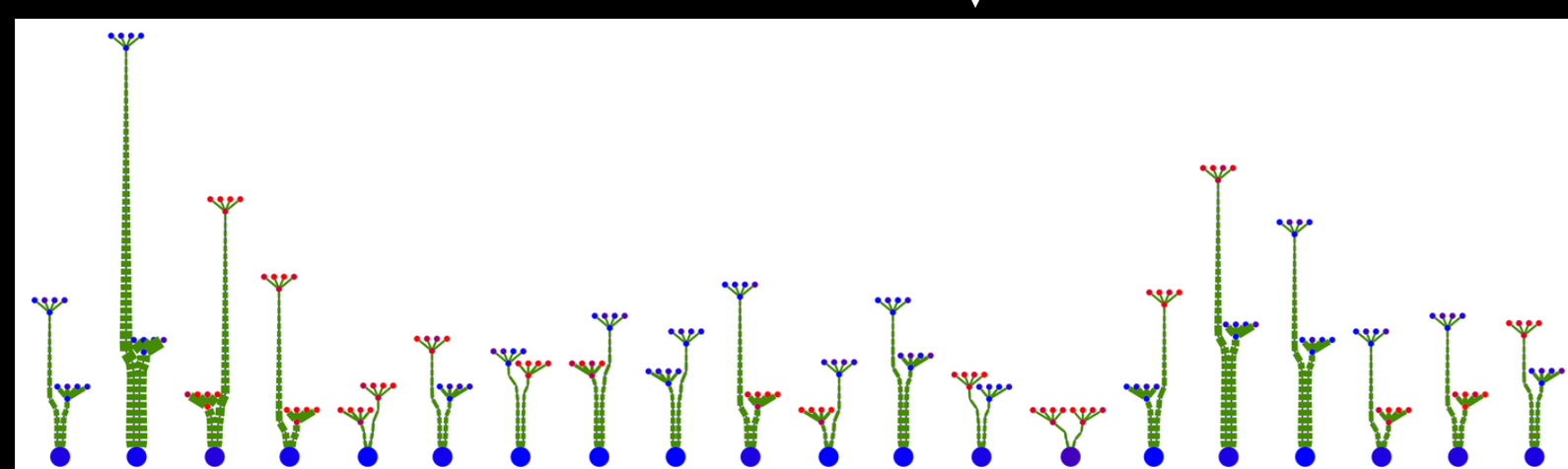
Posterior samples:



Example



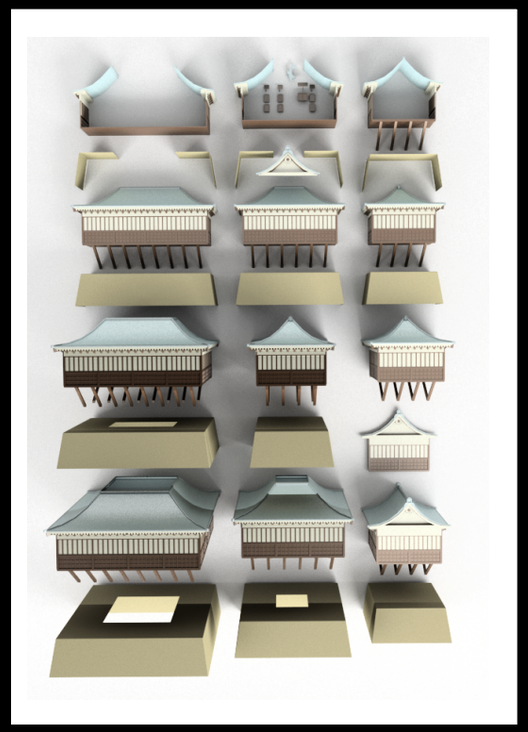
```
(begin (define F4 (lambda (V7 V8) (node (F1 V7 0.1) V8)))
    (define F3 (lambda (V6) (node (F1 V6 0.3))))
    (define F2
      (lambda (V3 V4)
        ((lambda (V5) (F4 V3 (F4 V4 V5)))
          (if (flip 9/11)
              (F2 81.0 85.0)
              (uniform-choice
                     (node (F1 204.0 0.3) (F3 199.0)
                       (F3 243.0) (F3 233.0) (F3 240.0)))
                     (F4 151.0
                       (node (F1 -21.0 0.3) (F3 7.0)
                         (F3 49.0) (F3 3e1) (F3 44.0)))))))
    (define F1
      (lambda (V1 V2)
        (data (color (gaussian V1 25)) (size V2))))
    (lambda ()
      (uniform-choice
             (node (F1 13.0 1) (F2 89.0 111.0)
               (F2 85.0 121.0))))
```



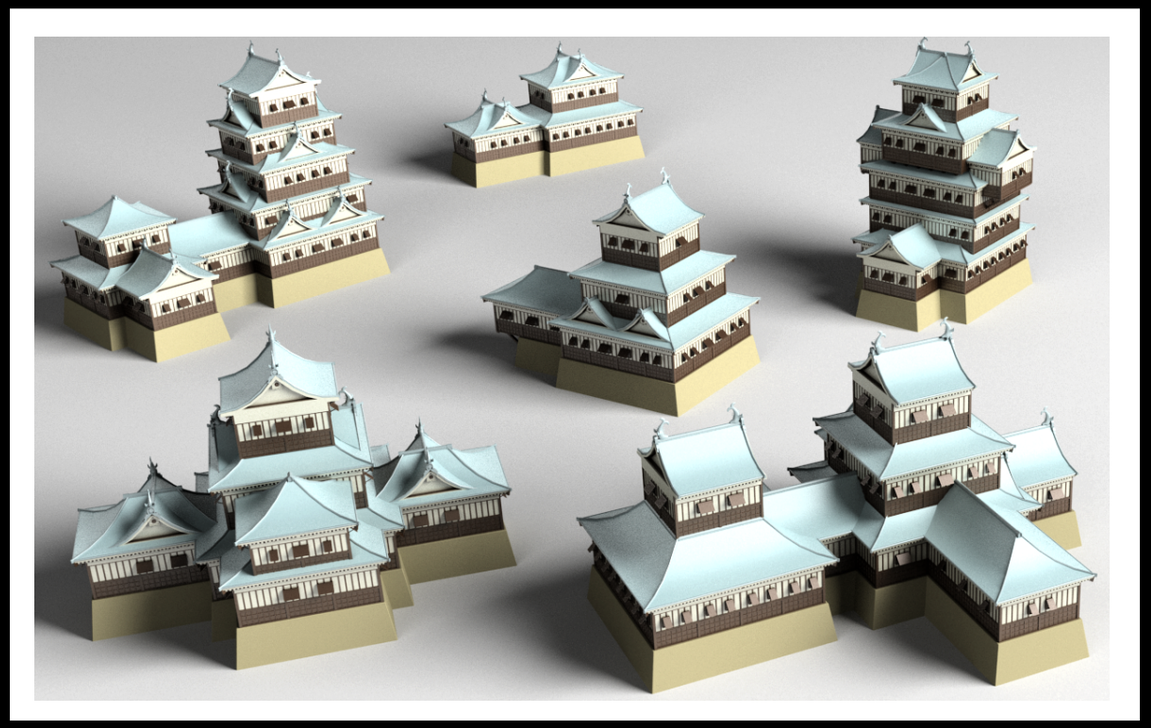
Application

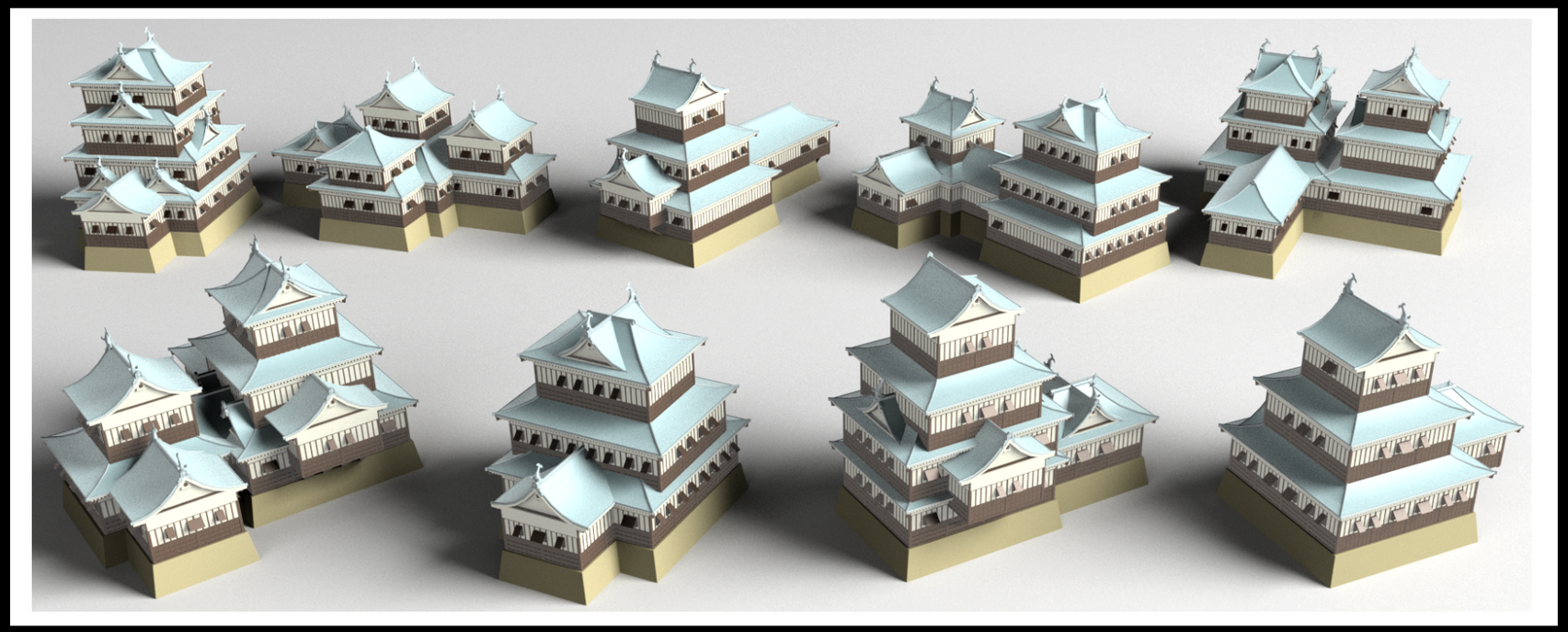
- **Inverse** procedural modeling:
  - Procedural model induced from a small set of artist-created examples.
  - Jerry Talton (Talton, et al, in prep.)

Terminals



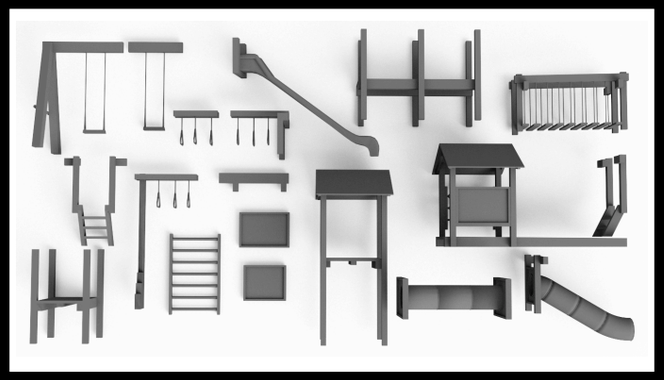
Examples

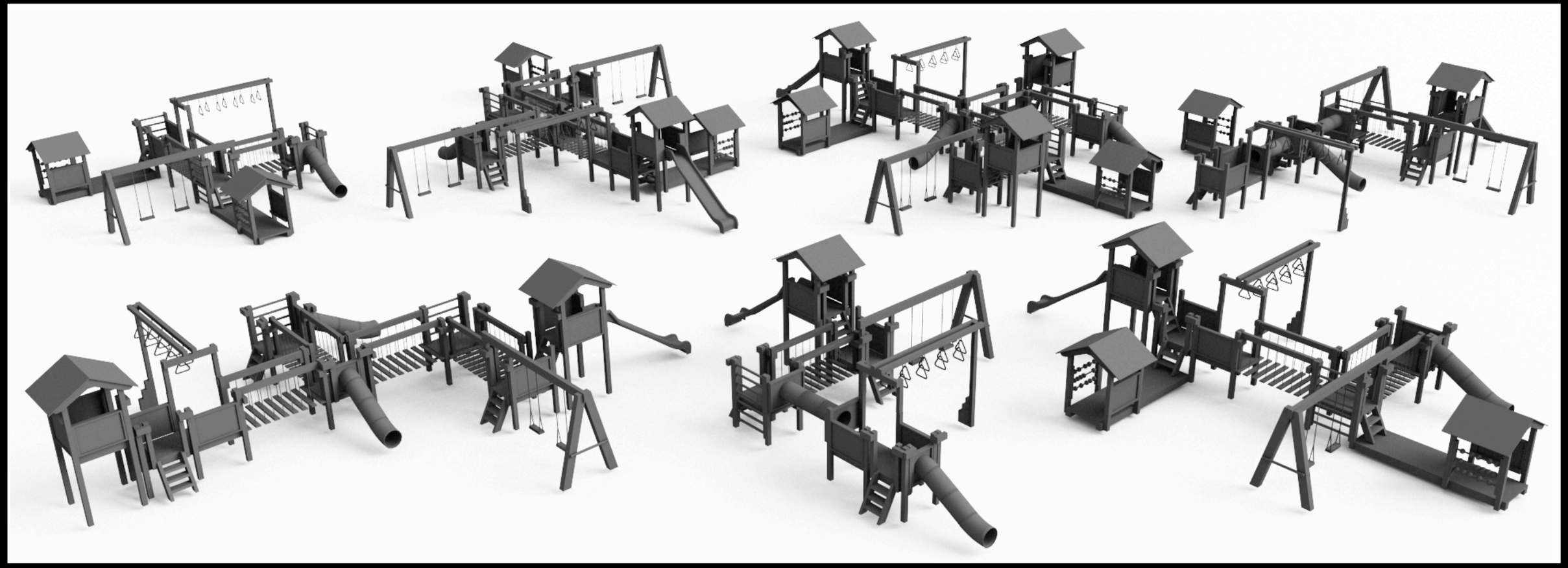




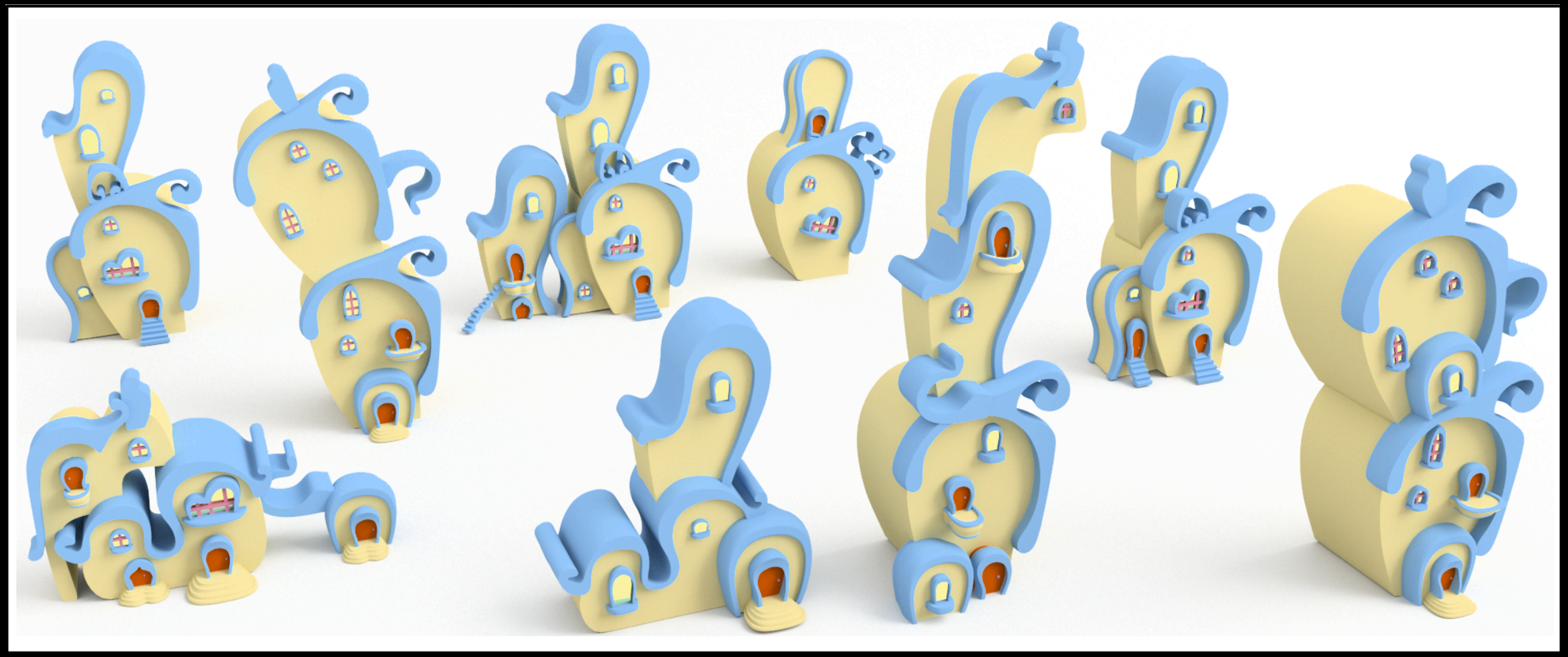
Posterior samples

Application





Application



Conclusion

Probabilistic inference

Generative models

Compositional representations

Probabilistic language of thought